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THE

FRANKLIN

ELEMENTARY ALGEBRA

BY

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PREFACE.

THE method of teaching algebra set forth in this book assumes as a leading principle that algebraic language, like other language, is best acquired, not through definitions and formal rules, but rather through actual use of the language itself as an instrument of thought. The singular power of algebraic language in analysis and reasoning is understood only when one has learned to use it in those processes. The earlier a beginner can be taught to use even a little of this language in reasoning, the better; for so may his interest in it be earlier awakened through the conscious exercise of a new power.

Accordingly, the first part of this book is devoted to the solution of easy problems, by the use of simple equations, with one unknown quantity. The few symbols needed are briefly explained in the first section. The second section is a carefully arranged collection of problems, interspersed with explanations, as new features in the process of solution call for them. Small numbers are used in a portion of the problems, that the reasoning may with ease be carried on mentally and expressed orally. This oral use of algebraic language is found to be a very efficient means of instruction.

Regarding the processes used in the solution of equations, the remark should be made here, that all these processes are based directly on the axioms given in Article 24. The process called "transposition" is not used, and the word "transpose" does not occur. It is believed to be better for the beginner to fix his thought directly on the real nature of the step he is taking than it is to run the risk of forming misconceptions through the use of terms which do not rightly mark the real nature of that step; for experience has shown that the operation called "transposition," though perfectly legitimate when properly understood, is, nevertheless, likely to be misleading and confusing to the beginner. Throughout this book, accordingly, equations are reduced by direct applications of the axioms, that is, the same quantity is added to or subtracted from each member, or each member is multiplied or divided by the same quantity.

The third section is devoted to operations on algebraic quantities. The matter given in this section is more commonly given at the beginning of text-books; but the arrangement here adopted recognizes the principle that the meaning of algebraic expressions must be clearly apprehended, before rules for operating upon or with them can be well understood. Attention is asked to the manner in which negative numbers are introduced, to the distinction made between algebraic and arithmetical numbers, and to the distinction between algebraic addition, subtraction, multiplication, and division, and the arithmetical operations of the same names. All the algebraic operations are based on the nature of algebraic numbers as distinguished from arithmetical numbers.

The learner is not in full possession of the algebraic method of reasoning until he has learned how to solve problems in a general manner, and to interpret the results for particular cases. The sixth section contains some illustrations of this process of generalization applied to problems already familiar. A degree of familiarity with generalized statements and processes, on the part of the learner, is assumed throughout the rest of the book. He is supposed to be ready to use the general language of algebra in the investigation of such topics as the binomial theorem, roots of numbers, arithmetical and geometrical progression, and logarithms. He is taught to guide his processes, not by rules, but by formulas, as in the solution of binomial equations in the ninth section, of quadratic equations in the tenth, and in various other instances. All this is in pursuance of the principal aim of algebraic training, which is to enlarge and generalize our notions of number and our methods of reasoning on numerical questions.

The course of study embraced in this book is sufficient to meet the requirements for admission to any of our colleges, and is such as is pursued in our best high schools and academies. The work is submitted to teachers in the hope that it may prove helpful in promoting improved methods of mathematical instruction.

BOSTON, May, 1882.

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ELEMENTARY ALGEBRA.

SECTION I.

THE LANGUAGE OF ALGEBRA.

ARTICLE **1.** Algebra is a method of reasoning about numbers in a language of symbols.

Two kinds of symbols are used, symbols of quantity and symbols of operation.

2. Symbols of quantity are letters of some alphabet, which are used to represent numbers known or unknown.

3. Symbols of operation are certain signs used to denote mathematical operations.

4. The sign + denotes addition, and is called ***plus.***

Thus, the addition of 5 to 8 is denoted by $8 + 5$, and the expression is read "eight plus five." So, too, if a stands for some number, and x for some other number, the sum of these numbers is denoted by $a + x$, and the expression is read "a plus x."

5. The sign − denotes subtraction, and is called ***minus.***

Thus, the subtraction of 5 from 8 is denoted by $8 - 5$, and the expression is read "eight minus five." So, too, the subtraction of the number x from the number a is denoted by $a - x$, and the expression is read "a minus x."

6. The sign × denotes multiplication, and is called the ***sign of multiplication.***

Thus, the multiplication of 8 by 5 is denoted by 8 × 5, and the expression is read "eight multiplied by five." So, too, the multiplication of the number a by the number x may be denoted by $a \times x$. It is usual, however, to omit the sign of multiplication when one or both the factors are represented by letters. Thus, instead of $a \times x$, we write $a x$; and, instead of $8 \times x$, we write $8 x$; but instead of 8 × 5, we should not write 85, because the expression 85 always retains the meaning given to it in arithmetic, the sum of 8 tens and 5 units.

The expression $a x$ may be read "a multiplied by x; but it is usual to omit the words "multiplied by," and say merely "$a x$."

7. The signs ÷ and : denote division, and are called ***signs of division.***

Thus, the division of 8 by 4 is denoted by 8 ÷ 4, or by 8 : 4, and either expression is read "eight divided by four." So, too, the division of the number a by the number x is denoted by $a \div x$, or by $a : x$, and either expression is read "a divided by x."

Another way of denoting division is to *write the dividend as numerator, and the divisor as denominator, of a fraction.* Thus, the division of 8 by 4 is denoted by $\frac{8}{4}$, which is read "eight divided by four;" and the division of a by x is denoted by $\frac{a}{x}$, which is read "a divided by x."

8. A number represented by an algebraic symbol or combination of symbols is an ***algebraic quantity.***

9. Oral Exercises.

1. Suppose a to represent the number of pigeons on one tree, and b the number on another tree. What will represent the number on both trees?

2. Suppose a to stand for the number of pears taken from one basket, b for the number taken from another basket, and c for the number taken from another. What will stand for the whole number taken?

3. A carriage costs eighty dollars more than a horse. If x represents the cost of the horse, what will represent the cost of the carriage?

4. If x represents the number of cents you paid for a slate, and you paid as much more for a book, what will represent the number of cents you paid for both?

10. Since the number represented by $x + x$ is twice as great as that represented by x, the quantity $x + x$ may be written as a product, $2x$, and read "two x." For a similar reason $3x$ is written instead of $x + x + x$, $4x$ instead of $x + x + x + x$, and so on.

5. Three times x and five times x are how many times x? What, then, may be written instead of $3x + 5x$?

6. What may be written instead of $x + 3x + 7x$?

7. If x stands for the number of cents that must be paid for a pound of coffee, what will stand for the cost of seven pounds?

8. Represent the cost of five quarts of cherries at x cents a quart.

9. If a represents the number of books on a shelf, and I take down a number of them represented by b, what will represent the number that remain?

10. The difference between two numbers is three. If x stands for the larger, by what may the smaller be represented?

11. The sum of two numbers is twenty-five. If one of them is represented by x, what will represent the other?

12. If three times a certain number be subtracted from seven times that number, how many times that number will remain? What, then, may be written instead of $7x - 3x$?

13. What may be written instead of $9x + 3x - 5x$?

14. George has seven cents less than William. If x stands for the number of cents William has, what will represent the number of cents George has? What will represent the number both have?

15. What is the remainder when three times x and five

times x are subtracted from twelve times x? What, then, may be written instead of $12x - 3x - 5x$?

16. What may be written instead of $5x + 3x - 2x + x$?

11. The quantity $5x + 3x - 2x + x$ is made up of four parts which are added or subtracted as indicated by the signs + and −. The parts of an algebraic quantity which are combined by addition or subtraction are called ***terms.***

Finding a single term equivalent to two or more separate terms is called ***uniting terms.***

17. Unite the terms of $7x + 9x + 6x - 14x$.

18. Unite the terms of $3y + y - y - y - y$.

12. Only like terms can be united. In the quantity $3x + 5x + 4 + 6$ the first two terms are alike, each being a certain number of times x; the last two terms are alike, each being a certain number of times 1; but the first two are unlike the last two. Hence the shortest expression for $3x + 5x + 4 + 6$ is $8x + 10$.

13. The order in which terms are united is of no consequence.

Thus, $10 - 3 + 10 - 6$ and $10 + 10 - 3 - 6$ have the same value, that is 11; and $x - 3 + x - 6$ and $x + x - 3 - 6$ have the same value, that is $2x - 9$.

Unite terms in the following quantities:

19. $2x + 10 + 4x + 7$.
20. $5x + 6 - 3x - 3$.
21. $3x + 3 + 4x + 4 - 5x - 5$.
22. $5x - 8x + 12x$.
23. $-5 + 4x - 3 + 2x$.
24. $-3x - 3 + 5x + 6$.
25. $y - \frac{1}{2} + y - \frac{1}{4} + y - \frac{1}{4}$.
26. $x - 1 + x - 1 + x - 1 + x - 1$.

SECTION II.

SIMPLE EQUATIONS.

14. The sign = stands for the words "equals," or "is equal to," and is called the ***sign of equality.***

15. An algebraic expression in which the sign of equality occurs is an ***equation.*** Thus, $7 + 5 = 4 \times 3$ is an equation.

An equation is a sentence asserting that whatever is written on one side of the sign of equality equals whatever is written on the other side. Thus, $7 + 5 = 4 \times 3$ is equivalent to the sentence, *The sum of seven and five equals the product of four and three.*

16. The quantity written at the left of the sign of equality is the ***first,*** or ***left-hand member*** of the equation, and that written at the right of the sign is the ***second,*** or ***right-hand member.***

Unknown terms in the first member: known terms in the second.

17. Oral Exercises.

1. If $3x = 6$, what number does x stand for?
2. If $2x = 8$, what number does x stand for?
3. If $7x = 28$, what number does x stand for?
4. What number is represented by x when $4x = 20$?
5. What number is represented by y when $6y = 42$?

The number represented by a letter is called the ***value*** of that letter.

6. What is the value of x when $5x = 30$?
7. What is the value of y when $8y = 56$?
8. Given $x + 3x = 8$, to find the value of x.

9. Given $z + 5z = 54$, to find the value of z.
10. Find the value of x in $x + 5x + 4x = 30$.
11. Find the value of x in $3x + 8x = 77$.

18. ILLUSTRATIVE PROBLEM. A boy bought a peach and a pear for eight cents, paying for the pear three times as much as for the peach. How much did he pay for each?

The easy problems given among the oral exercises are suited to oral work; but the beginner will do well to write down some of his oral solutions in the form here given.

Solution. Although the price of the peach is not known, we may let some letter represent it, and then we can reason about it just as well as if it were known.

Let x represent the number of cents he paid for the peach. Then $3x$ will represent the number of cents he paid for the pear; and the number of cents he paid for both will be represented by $x + 3x$.

WRITTEN WORK.

x = price of the peach.
$3x$ = price of the pear.

$$x + 3x = 8$$
$$4x = 8$$
$$x = 2.$$

Now, we know that the number of cents he paid for both is 8. To state this fact in algebraic language, we say $x + 3x = 8$. By uniting terms, this equation becomes $4x = 8$, from which we see that x must represent 2.

Ans. The peach cost 2 cents, and the pear 6 cents.

The quantity represented by x in the solution of a problem is called the ***unknown quantity.***

19. Oral Exercises: Problems.

12. A bat and a ball together cost sixty cents, and the ball costs four times as much as the bat. Find the cost of each.

13. Charles bought a book and a slate for seventy-five cents, paying for the book twice as much as for the slate. What did he pay for each?

14. For fifty-four cents Mary bought some worsted and some muslin, paying for the former five times as much as for the latter. How much did she pay for each?

15. A pedestrian walked a certain distance before breakfast, five times as far during the forenoon, and four times as far during the afternoon. The whole day's journey was thirty miles. How far did he walk in each part of the day?

20. ILLUSTRATIVE PROBLEM. Albert bought oranges at three cents apiece, and the same number of bananas at eight cents apiece, paying for the whole seventy-seven cents. How many of each did he buy?

Solution. Let x represent the number of each. Then x oranges at 3 cents apiece will cost $3x$ cents, and x bananas at 8 cents apiece will cost $8x$ cents. The equation, therefore, is $3x + 8x = 77$, from which we find the value of x to be 7.

Ans. He bought 7 oranges and 7 bananas.

WRITTEN WORK.

$x =$ number of each.

$3x =$ cost of the oranges.

$8x =$ cost of the bananas.

$$3x + 8x = 77$$

$$11x = 77$$

$$x = 7.$$

Proof. To test the correctness of this result we must find whether 7 oranges at 3 cents apiece and 7 bananas at 8 cents apiece would cost 77 cents, as stated in the problem.

16. A farmer bought an equal number of sheep and calves, the sheep at three dollars apiece, and the calves at five dollars apiece, paying for the whole seventy-two dollars. How many of each did he buy?

17. Ellen bought some 3-cent and some 2-cent postage-stamps, of each the same number, paying for the whole fifty cents. How many stamps of each kind did she buy?

18. Robert bought some peaches at three cents apiece, and twice as many pears at four cents apiece, paying for the whole thirty-three cents. How many of each did he buy?

Partial Solution. Let x represent the number of peaches; then $2x$ will represent the number of pears. Now x peaches at 3 cents apiece will cost $3x$ cents; and $2x$ pears at 4 cents apiece will cost $8x$ cents.

19. Kate bought some red ribbon at six cents a yard, and twice as much blue ribbon at seven cents a yard, paying for

the whole eighty cents. How many yards of each kind did she buy?

20. A traveller rode a certain length of time at the rate of ten miles an hour, and walked four times as long at the rate of three miles an hour, to finish his journey of forty-four miles. How long did he ride, and how long did he walk?

21. John has twice as much money as William, and Edgar has twice as much as they both, while all three together have ninety cents. How many cents has each?

22. In a certain school there are forty-eight pupils, and there are twice as many girls as boys. How many of each are there?

23. Divide thirty-six nuts between two boys so as to give three times as many to one boy as to the other.

24. A pole stands partly in the mud, partly in the water, and partly in the air. The part in the air is as long as the parts in the mud and water together; the part in the water is five times as long as the part in the mud; and the whole length of the pole is twenty-four feet. What is the length of each part?

25. Two travellers set out at the same time from towns fifty-six miles apart, and walked towards each other till they met, one at the rate of three miles, and the other at the rate of four miles, an hour. How long had they been walking when they met?

26. Three times a certain number added to four times the double of that number gives for the sum fifty-five. What is the number?

21. **Written Exercises: Problems.**

27. A horse and a harness are together worth $315, but the horse is worth six times as much as the harness. What is the value of each?

28. Separate the number 378 into two parts, such that one part shall be thirteen times as great as the other.

29. The sum of two numbers is 1504, and one is forty-six times as great as the other. What are the numbers?

30. A debtor made a payment of 5200 dollars to discharge his debt, both principal and interest. The principal was twenty-five times as much as the interest. What was the principal? and what was the interest?

31. Find two numbers such that one is eighteen times as large as the other, the difference between them being 408.

32. A grocer sold tea at 50¢ a pound, and coffee at 20¢ a pound, of each the same quantity, getting for the whole $8.40. How many pounds of each did he sell?

33. A clerk went to a bank with a check for $36, and asked to have for it half-dollars, quarters, dimes, and half-dimes, of each the same number. What was the number?

34. A carpenter, 12 journeymen, and 4 apprentices, receive for a certain time $198 as wages. The carpenter receives two dollars a day; each journeyman, a dollar and a half; and each apprentice, half a dollar. How many days have they worked for this money?

35. A courier rode a certain number of hours at the rate of 12 miles an hour, and then, to finish his journey of 286 miles, rode as much longer at the rate of 14 miles an hour. How long was he in making the whole journey?

36. A farmer paid for a farm and the buildings $14400. The house was valued at three times as much as the barn, and the land at twice as much as the house and barn together. What was the value of each?

37. A man dying leaves by his will $11000 to be divided between his widow, two sons, and three daughters. He intends that the widow shall receive double the share of a son, and that each son shall receive twice as much as a daughter. How much will each receive?

38. Mary bought some apples at 1¢ apiece, oranges at 3¢ apiece, and bananas at 9¢ apiece, paying for the whole $1.92. There were three times as many apples as oranges, and half as many bananas as apples and oranges together. How many were there of each?

39. A mixture made of corn worth 80¢ a bushel, oats

worth 45¢ a bushel, and barley worth $1.10 a bushel, is worth $97.20 in all. There is twice as much corn as barley, and twice as much oats as corn and barley together. How many bushels are there of each?

40. Twenty-seven times a certain number added to seventeen times the double of that number gives 2989 for the sum. What is the number?

41. The difference between thirty-seven times a certain number and nine times the double of that number is 1007. What is the number?

42. Divide 275 acres of land among A, B, and C, in such a way that A may have twice as much as B, and B thrice as much as C.

43. In a certain factory the wages of the men are $1.50 a day, of the women 90¢, and of the children 40¢; and the whole amount paid the operatives for a week's work is $693.60. There are five times as many women as men, and one-third as many children as men and women together. How many operatives of each class are there?

44. Four men, A, B, C, and D, agreed to contribute $6600 for founding a public library in their native town. B gave twice as much as A, C as much as A and B together, and D as much as B and C. How much did each contribute?

22. By reviewing the work of the foregoing problems, it may be seen that when a problem is solved by algebra there are two things to be done:

(*a*.) To express the statements of the problem in algebraic language. This is called ***putting the problem into an equation.***

(*b*.) To reduce the equation so as to find the value of the unknown quantity. This is called ***solving the equation.***

For putting problems into equations no definite rules can be given. It is a matter of skill, which, like skill in any other art, is gained by practice. In solving equations, however, we are guided by certain general principles called axioms.

23. An ***axiom*** is a statement that needs no proof: it is self-evident; that is, no one, if he understands the words, can fail to accept the statement as true.

24. Axioms.

1. If the same quantity or equal quantities be added to equal quantities, the sums will be equal.

2. If the same quantity or equal quantities be subtracted from equal quantities, the remainders will be equal.

3. If equal quantities be multiplied by the same quantity or by equal quantities, the products will be equal.

4. If equal quantities be divided by the same quantity or by equal quantities, the quotients will be equal.

5. The value of a quantity is not changed by both adding to it and subtracting from it the same quantity.

6. If the product of two quantities be divided by either of them, the quotient will be the other.

7. Two quantities each equal to a third quantity are equal to each other.

25. Exercises illustrating the Axioms.

a. What are the results if each member of the equation $4x = 20$ be divided by 4? Are these results equal? Why? (Axiom 4.)

b. If it is true that $3x = 24$, why does it follow that $x = 8$?

c. What are the results if 4 be subtracted from each member of the equation $2x + 4 = 14$? Are these results equal? Why? (Ax. 2.) What equation, then, may be written?

d. If it is true that $x+7=17$, am I right in saying that $x=10$? Why?

e. A man had a certain number of dollars in his pocket. During the day he received 3 dollars, and spent 3 dollars. How much money had he then? (Ax. 5.)

f. What is the quantity $x+3-3$ equivalent to?

g. A man, having a certain number of dollars in his pocket, paid out 4 dollars, and afterwards received 4 dollars. How much money had he then?

h. What is the quantity $x-4+4$ equivalent to?

i. If 4 be added to each member of the equation

$$3x-4=11,$$

what equation will be obtained? Why? (Ax. 1.)

j. How may the first member of the equation

$$3x-4+4=15$$

be reduced? Why? (Ax. 5.)

k. What is the quantity $\frac{3x\times 4}{4}$ equivalent to? Why? (Ax. 6.)

l. What are the results if each member of the equation $\frac{3x}{4}=6$ is multiplied by 4? Do these results form an equation? Why? (Ax. 3.)

m. If it is true that $\frac{x}{5}=7$, does it follow that $x=35$? Why?

n. How do you know that $3x=24$ when $\frac{3x}{4}=6$?

o. What equation may be derived from $\frac{2x}{3}+7=13$ by aid of Ax. 2?

p. What equation may be derived from $\frac{2x}{3}=6$ by aid of Ax. 3?

q. What equation may be derived from $2x=18$ by aid of Ax. 4?

Unknown Terms in the First Member: Known Terms in Both Members.

26. Oral Exercises.

45. If $x + 2 = 6$, what number does x stand for?

46. If $2x + 3 = 11$, what does $2x$ equal? If $2x = 8$, what is the value of x?

47. If $3x + 7 = 16$, what is the value of x?

48. What is the value of x when $5x + 7 = 17$?

Solve the following equations:

49. $3x + 5 = 20$.
50. $5x + 5 = 30$.
51. $x + 18 = 38$.
52. $8x + 1 = 9$.
53. $7x + 3 = 24$.
54. $6x + 3 = 42$.

55. If $x - 2 = 6$, what number does x stand for?

56. If $2x - 3 = 7$, what does $2x$ equal? What is the value of x?

57. If $2x - 5 = 9$, what is the value of x?

58. What is the value of x when $5x - 4 = 11$?

Solve the following equations:

59. $3x - 11 = 10$.
60. $7x - 8 = 20$.
61. $6x - 3 = 15$.
62. $3x - 9 = 30$.
63. $8x - 5 = 3$.
64. $4x - 17 = 3$.

27. From these exercises it will be seen that, when an equation has known terms in both members, we *subtract from or add to each member such known terms as will leave one member free from known terms.*

28. Oral Exercises: Problems.

65. George had 20 cents. He bought some oranges at 3 cents apiece, and gave the rest of his money, which was 5 cents, to a beggar. If x represents the number of oranges he bought, what will represent their cost? What will $3x + 5$ represent? What equation, then, may be formed? How many oranges did he buy?

66. A courier, after riding a certain number of hours at the rate of 9 miles an hour, found that he must ride 7 miles more

to finish his journey of 70 miles. How many hours had he been riding?

67. William and John on counting their money find that together they have 40 cents, but William has 10 cents more than John. If x represents the number of cents John has, what will represent the number William has? What will represent the amount both have? What equation may be formed? Solve the equation $x+x+10=40$.

68. Mary is 5 years older than her brother, and their united ages amount to 27 years. What is Mary's age?

69. In example 67, if x represent the number of cents William has, what will represent the number John has? What equation may be formed? Solve the equation $x+x-10=40$.

70. Henry is 4 years younger than his sister, and their united ages amount to 30 years. What is Henry's age?

29. ILLUSTRATIVE PROBLEM. Arthur and Walter had equal sums of money. When Arthur had spent 8 cents and Walter 9 cents, they put what they had left together, and found that it amounted to 31 cents. How many cents had each at first?

Solution. Let x represent the number of cents each had at first; then $x-8$ will represent what Arthur had left, and $x-9$ what Walter had left. The sum of these remainders is $x-8+x-9$, which, by the statement of the question, is 31 cents. Hence the equation $x-8+x-9=31$.

Ans. Each had 24 cents.

WRITTEN WORK.

$x =$ number of cents each had.

$x-8 =$ number Arthur had left.

$x-9 =$ number Walter had left.

$$x-8+x-9=31$$
$$2x-17=31$$
$$2x=48$$
$$x=24.$$

71. There were two purses, each containing the same number of coins. When 3 coins had been taken from one purse and 5 coins from the other, there were found to be 16 coins remaining in both purses together. How many coins were there in each purse at first?

72. John is twice as old as Margaret, and 4 years ago their united ages amounted to 19 years. What are their ages now?

73. What is the sum of $x-4$ and $x-5$?

74. What is the sum of $x-3$ and $x-4$ and $x-6$?

75. What is the sum of $x-2$ and $x-2$ and $x-2$?

It will be noticed that the quantity $x-2$ is taken 3 times to form the sum just found. Hence $3x-6$ is the *product* of the quantity $x-2$ multiplied by 3. This product may be found by multiplying each term of $x-2$ separately by 3.

76. What is the sum of $x-3$ and $x-3$ and $x-3$ and $x-3$? What, then, is the product of $x-3$ multiplied by 4? Show how this product may be obtained by multiplying the terms separately.

What is the product

77. Of $x-3$ by 5?

78. Of $x-7$ by 4?

79. Of $2x-3$ by 6?

80. Of $x+7$ by 3?

81. Of $2x-8$ by 5?

82. Of $3x+5$ by 10?

30. Illustrative Problem. A grocer sold 6 pounds of sugar and 4 pounds of rice for 70 cents, asking 5 cents less for a pound of rice than for a pound of sugar. What was the price of each?

Solution. Let x represent the price of a pound of sugar; then $x-5$ will represent the price of a pound of rice. Six pounds of sugar at x cents a pound will cost $6x$ cents; and four pounds of rice at $x-5$ cents a pound will cost $4x-20$ cents. The equation, therefore, is $6x+4x-20=70$.

83. A fruit-vender sold 7 oranges and 4 pears for 47 cents, and the price of a pear was 2 cents less than the price of an orange. What was the price of each?

84. The difference between two numbers is 7, and the result obtained by adding three times the less to four times the greater is 42. What are the numbers?

85. At 6 o'clock A.M., a man sets out to walk from A to B, a distance of 40 miles, and he walks at the rate of 3

miles an hour. At 10 o'clock A.M. of the same day another man sets out from B and walks towards A, at the rate of 4 miles an hour. How many hours will each have been walking when they meet?

86. A cashier paid out $26 in one-dollar and two-dollar bank-notes, and the number of notes of the latter kind was less than the number of the former kind by 5. How many notes of each kind did he use?

87. A pound of coffee costs 5 cents less than four pounds of sugar, and I can buy 2 pounds of coffee and 9 pounds of sugar for $1.60. What are the prices of each?

88. Charles had three times as many marbles as Ralph, but, after he had lost 7 and Ralph had bought 9, they put what they had together, and found they had 54. How many had each at first?

31. Written Exercises: Problems.

89. Two merchants, A and B, charter a ship for $1125, of which A pays $325 more than B. How much does each pay?

90. At a certain election 1297 ballots were cast for two candidates, and the winning candidate had a majority of 125. How many had each?

91. Separate the number 3957 into two parts, making one part 335 greater than the other part.

92. Separate the number 4987 into two parts, making one part 295 less than the other part.

93. How far has a ship sailed on her voyage of 2890 miles when the remainder of the voyage is reckoned to be 154 miles less than the distance already sailed?

94. In a certain election 5789 votes were cast for three candidates. One of the losing candidates received 593 votes less, and the other 2114 votes less, than the winning candidate. How many votes did each receive?

95. Divide a line 33 feet long into four parts, of which the second shall be $1\frac{1}{2}$ feet longer than the first, the third $2\frac{1}{2}$ feet longer than the second, and the fourth $3\frac{1}{2}$ feet longer than the third.

96. A boy has 155 hills of potatoes to dig in five hours. He proposes to lay out his work in such a way that he may have each hour six hills fewer to dig than in the hour preceding. To do this how many hills must he dig in each of the five hours?

97. Divide a pole 12 feet 3 inches long into seven parts such that each part shall be two inches longer than the part next above it.

98. It is required to divide the number 99 into five such parts that the first may exceed the second by 3, be less than the third by 10, greater than the fourth by 9, and less than the fifth by 16.

99. A merchant tailor bought 12 pieces of cloth for $700. Two pieces were white, three black, and seven blue. A piece of the black costs $10 more than a piece of the white, and a piece of the blue costs $15 more than a piece of the black. Required the price of each kind.

100. A cistern which holds 820 gallons is filled in 20 minutes by three pipes, one of which carries 10 gallons more, and another 5 gallons less, per minute, than the third. How much flows through each pipe per minute?

101. A man by working 9 days, and having the help of his son 5 days, does a piece of work for which he is paid $13.75. The wages of the son are 75¢ less a day than the wages of the father. What are the day's wages of each?

102. A farmer rents 124 acres of land for $173.50. The land is of two sorts, the better being rented at $1.75 an acre, and the poorer at $1.25. How many acres of each sort are there?

103. A pile of 60 cannon-balls weighs 732 pounds. The balls are of two sizes, some weighing 10 pounds, and others 16 pounds apiece. How many of each size are there in the pile?

104. A grocer has two kinds of tea; one worth 35¢ a pound, and the other 42¢ a pound. He wishes to make a mixture of 50 pounds in such a way that he may sell it without gain or loss for 38¢ a pound. How many pounds of each kind must he use in the mixture?

105. A debt of $1265 was paid in bank-notes, 5's, 10's, and 20's. There were used 3 less than twice as many 5's as 10's, and 12 more than thrice as many 20's as 10's. How many of each denomination were used?

106. A father bequeathes $8600 to his four children. According to the will, the share of the eldest is to be $100 less than double that of the second; the share of the second, $200 less than thrice that of the third; and the share of the third, $300 less than four times that of the fourth. What is the share of each?

Fractional Unknown Terms in the First Member: Known Terms in Both Members.

32. Oral Exercises.

107. If $\frac{x}{2} = 5$, what is the value of x?

108. If $\frac{x}{3} = 4$, what is the value of x?

109. If $\frac{3x}{5} = 6$, what is the value of $3x$? Of x?

110. What is the product of $\frac{3x}{5}$ multiplied by 5?

111. What is the product of $\frac{7x}{12}$ multiplied by 12?

112. What is the product of any fraction multiplied by its denominator?

113. What is the product of $\frac{3x}{4}$ multiplied by 12?

NOTE. First multiply by 4, then multiply the result by 3.

114. What is the product of $\frac{4x}{5}$ multiplied by 15?

115. What is the product of $\frac{x}{2} + \frac{2x}{3}$ multiplied by 6?

116. Multiply $\frac{x}{3} + \frac{3x}{4}$ by 12.

117. Multiply $\frac{2x}{3} - \frac{4x}{9}$ by 9.

118. Multiply $x + \frac{2x}{3} - \frac{x}{2}$ by 6.

119. What equation is obtained when both members of $x + \frac{x}{4} = 15$ are multiplied by 4?

120. What equation is obtained when both members of $2x - \frac{3x}{5} = 14$ are multiplied by 5?

121. What equation is obtained when both members of $\frac{x}{3} + \frac{x}{4} = 7$ are multiplied by 3?

122. What equation is obtained when both members of $x + \frac{3x}{4} = 21$ are multiplied by 4?

123. What equation is obtained when both members of $\frac{x}{3} + \frac{x}{4} = 7$ are multiplied by 12?

33. The results in the last two exercises are equations in which no fractional terms occur. In such cases it is said that the given equation has been ***cleared of fractions.***

To clear an equation of fractions: *Multiply each member of the given equation by one of the denominators, then each member of the result by another denominator, and so on, till an equation is obtained free from fractions.*

Or, *Multiply each member of the given equation by the least common multiple of all the denominators.*

Clear of fractions, and then solve the following:

124. $2x + \frac{3x}{4} = 11.$

126. $\frac{3x}{7} = 6.$

125. $3x - \frac{3x}{4} = 9.$

127. $\frac{2x}{3} + \frac{5x}{6} = 9.$

128. $\frac{x}{2}+\frac{2x}{3}+9=30.$

129. $\frac{x}{2}-\frac{x}{5}=16.$

130. $\frac{x}{6}+\frac{x}{3}=1.$

131. $\frac{3x}{4}-\frac{3x}{5}=13\frac{1}{2}.$

132. $\frac{5x}{6}-\frac{x}{3}-1=2.$

133. $\frac{3x}{4}-\frac{x}{8}=1.$

34. **Oral Exercises: Problems.**

134. In a certain school there are fifty-five pupils, and there are five-sixths as many boys as girls. How many of each are there in the school?

135. I sold a cow for $45, thereby gaining one-fourth of what she cost me. What did the cow cost?

136. A merchant was obliged to sell cloth at 70¢ a yard, thereby losing one-eighth of the cost. What did the cloth cost?

137. Divide twenty-one apples between two boys so that the younger may have three-fourths as many as the elder.

138. Divide a straight line 16 inches long into two parts that shall be to each other in the ratio of 3 to 5.

NOTE. This means that one part shall equal $\frac{3}{5}$ of the other part.

139. Divide a straight line 24 inches long into three parts that shall be to each other in the ratio of 2, 3, and 7.

NOTE. This means that the first part shall equal $\frac{2}{7}$, and the second $\frac{3}{7}$, of the third part.

140. Charles spent one-third of his money for a book, and one-fourth of it for a slate, and these cost him 70 cents. How much money had he?

141. Two-thirds of a certain number added to five-sixths of it gives 9 for the sum. What is the number?

142. One-half and two-thirds of a certain number, and nine more, make thirty. What is the number?

143. James is five years older than his sister, but three-sevenths of her age added to his amounts to fifteen years What is the age of each?

144. What number increased by 20 per cent of itself amounts to 48?

NOTE. The expression 20 per cent, or, as it is often written, 20 %, means $\frac{20}{100}$, which, reduced to its lowest terms, is $\frac{1}{5}$.

145. What was the original length of a piece of cloth, which, after shrinking 15 per cent, measures 17 yards in length?

146. The difference between one-fourth and one-eighth of a certain number is one. What is the number?

147. Mary's father told her that the difference between three-fourths and two-thirds of his age was four years, and asked her to tell how old he was. What was the right answer?

148. Two travellers, A and B, set out on a journey at the same time from the same place. At the end of the first day A had gone seven-twelfths of the whole distance, B had gone two-thirds of it, and they were then seven miles apart. What was the whole distance to be travelled?

35. ILLUSTRATIVE PROBLEM. A can do a piece of work in five days, and B can do the same in six days. In how many days can they do it working together?

Solution. Let x represent the number of days in which both can do it working together. If A can do the work in five days, he will do one-fifth of it in a day, and in x days he will do x-fifths of it. Again, if B can do the work in six days, he will do one-sixth of it in a day, and x-sixths of it in x days. A and B together, in x days, will do $\frac{x}{5}+\frac{x}{6}$ of the work; and this, by the statement of the question, is the whole. Therefore the equation is

$$\frac{x}{5}+\frac{x}{6}=1,$$

from which we find that $x=2\frac{8}{11}$. *Ans.* In $2\frac{8}{11}$ days.

149. Two pipes discharge into a cistern. The first will fill the cistern in six hours, and the second in three hours. In how many hours will the cistern be filled by both pipes running together?

150. Ann can pare a basket of apples in eight hours, and

her mother can pare them in four hours. How long will it take them if they work together?

151. A bag of meal will last a horse ten days, or a cow fifteen days. How long will it last both?

152. A man can reap a field of rye in five days, and his son can reap it in twenty days. How long will it take them both working together?

153. Two travellers, A and B, set out at the same time from different towns, and ride towards each other. A's horse would travel the whole distance in four hours, and B's horse in five hours. In how many hours after they start do they meet?

154. A cistern is supplied by a pipe that would fill it in six hours, and discharged by a pipe that would empty it in eight hours. If, when the cistern is half full, both pipes are allowed to run, in how many hours will the cistern be filled?

155. How many days will it take A, B, and C together, to do a piece of work, if A can do it alone in four days, B in three days, and C in six days?

36. **Written Exercises: Problems.**

156. What number increased by three-eighths of itself will amount to 23463?

157. What number diminished by two-sevenths of itself will leave as a remainder 1765?

158. What number is that whose sixth part exceeds its seventh part by 116?

159. What number is that whose seventh part exceeds its ninth part by $\frac{1}{5}$?

160. What is the number, from the sixth part of which if six be taken, six times the remainder is equal to six?

161. How much money have I when the fourth part and the fifth part of it together amount to $38.25?

162. By selling cloth at 45 cents a yard, a merchant gains 25%. What did the cloth cost him?

163. The difference between two numbers is 1; and the

sum of five-eighths of the smaller and the whole of the larger is 40. What are the numbers?

164. A boy gave one-third of his chestnuts to his sister, sold one-fourth to his brother, ate one-sixth of them, and had thirty left. How many had he at first?

165. From a cask of oil one-twelfth of the whole had leaked out, and two-thirds had been drawn out, when there were found to be nine gallons left. How many gallons did the cask contain at first?

166. I bought some cloth at the rate of 7 dollars for 5 yards, and I sold the same at the rate of 11 dollars for 7 yards, thereby gaining $102. How much cloth was there?

167. What number increased by 25 per cent of itself will amount to 1725?

168. What number diminished by 5 per cent of itself will equal 8265?

169. What number increased by 13 per cent of itself will amount to 1000?

170. A debtor made a payment of $3887.25, which discharged the principal and interest of his debt. The interest was 0.095 of the principal. What was the principal?

NOTE. The decimal may be changed to a common fraction.

171. What sum of money, placed at interest for 1 year and 5 months, at 6 per cent, will amount to $400?

172. By the terms of a bequest a man will receive $8000 ten years hence, but he wishes to sell this claim now for its present value. The use of money being worth 5 per cent a year, simple interest, what is the present value of the claim?

173. If I am satisfied with an income of 5% on my investment, what can I afford to pay for a share of bank-stock that yields a yearly dividend of eight dollars?

174. A widow receives yearly an income of $705 from money at interest. Five-ninths of the money bears interest at the rate of 5%, and the rest at the rate of $5\frac{1}{2}$% a year. How much money has she at interest?

175. At what rate must $750 be placed at interest that it may yield $50 a year?

NOTE. The rate may be represented by $\frac{x}{100}$.

176. A market-man bought some chickens of one farmer at the rate of two for a dollar, and an equal number of another farmer at the rate of three for a dollar. He sold them all at the rate of five for two dollars, and lost two dollars by his trading. How many chickens did he buy?

177. A person can walk $3\frac{1}{2}$ miles an hour, or ride 9 miles an hour. How long will it take him to make a journey of 30 miles if he walks half the time, and rides half the time?

178. A student has just an hour and a half for exercise. He starts off on a horse-car which travels 10 miles an hour; and after a time he gets out, and walks home at the rate of 4 miles an hour. What is the greatest distance he can go from home so as to return within his time?

179. What distance has been gone over when the hind-wheel of a bicycle has made 600 revolutions more than the fore-wheel, the circumference of the latter being 14 feet, and that of the former $3\frac{1}{2}$ feet?

180. A party of girls walked to the top of a mountain at the rate of $2\frac{1}{4}$ miles an hour, and down the same path at the rate of $3\frac{1}{2}$ miles an hour, and they were walking 5 hours. How far did they walk?

181. A cistern can be filled by three different pipes; by the first in $1\frac{1}{3}$ hours, by the second in $3\frac{1}{3}$ hours, and by the third in 5 hours. In what time will this cistern be filled if all three pipes are opened at once?

182. A person can walk from one town to another in 14 hours, or ride the same distance in 6 hours. How long will it take him if he walks half the time, and rides half the time?

183. A bath-tub can be filled from the hot-water faucet in 10 minutes, from the cold-water faucet in 5 minutes, and emptied by the waste-pipe in 4 minutes. How long will it take to fill the bath-tub when both faucets are open and the waste-pipe closed? How long with the waste-pipe open?

184. Three men engaged to harvest a field of wheat, but they were unable to say beforehand how long it would take them. A knew that he alone could harvest the field in 6 days; B knew that he could do it in 8 days; and C knew that it would take him 10 days. How long would it take them working together?

Unknown Terms in Both Members: Known Terms in One or Both.

37. Oral Exercises.

185. From each member of the equation $3x = x + 6$, subtract x. Solve the resulting equation.

186. To each member of the equation $3x = 12 - x$, add x. Solve the resulting equation.

187. From each member of the equation $2x + 6 = 5x$, subtract $2x$. Solve the resulting equation.

NOTE. Having obtained the equation $6 = 3x$, we may interchange the members, and say $3x = 6$; since it is evidently unimportant which member is written first.

188. To each member of the equation $12 - 2x = 4x$, add $2x$. Solve the resulting equation.

38. From these exercises it will be seen, that, when an equation has unknown terms in both members, the first step in the solution is *to subtract from or add to each member such terms as will leave one member of the result free from unknown terms.*

What must be subtracted from or added to each member of the following equations, that one member may be free from unknown terms?

189. $4x = x + 12$.
190. $5x = 10 + 3x$.
191. $2x = 6 - x$.
192. $3x = 21 - 4x$.
193. $6 + 2x = 4x$.
194. $3x + 9 = 7x$.
195. $14 + 3x = 5x$.
196. $21 - 3x = 4x$.
197. $16 - x = 7x$.
198. $15 - 2x = 3x$.

Solve the following equations:

199. $3x + 7 = x + 19.$

200. $3x + 12 = 5x + 4.$

201. $x - 5 = 0.$

202. $5x + 10 = 8x - 5.$

203. $4x - 8 = 3x - 1.$

204. $7x - 21 = 0.$

205. $0 = 15 - 3x.$

206. $4x - 36 = 2x.$

207. $2x - 2 = \frac{x}{2} + 4.$

208. $2x - \frac{x}{3} = \frac{2x}{3} + 6.$

209. $\frac{x}{3} - 5 = 2 - \frac{x}{4}.$

210. $\frac{2x}{3} + 2 = \frac{3x}{5} + 3.$

39. Oral Exercises: Problems.

211. If fifteen be added to a certain number, the sum will be four times the number. What is the number?

212. What number will become seven times as great as itself by adding eighteen to it?

213. What number is diminished to two-fifths of itself by subtracting six from it?

214. A father is thirty years older than his son, and twice the father's age is equal to seven times the son's age. How old is each?

215. Said A to B, "I am eight years older than you." "Then," replied B, "five times your age is equal to seven times mine." How old was each?

216. A merchant sells broadcloth for one dollar a yard more than he sells doeskin for; but ten yards of doeskin would come to as much as eight yards of broadcloth. What is the price of each kind of cloth?

217. A's horse can travel as far in four days as B's can in five days, and the difference in one day is six miles. How far can each horse travel in a day?

218. A mother is now three times as old as her daughter, but ten years hence she will be only twice as old. What are their ages now?

219. A father is twenty-four years older than his son,

but ten years ago his age was five times that of his son. What are their ages?

220. A boy has a number of shells which he wishes to arrange in rows, with a certain number in each row. He finds that he can fill five rows, and have five shells left over; but to fill six rows he needs three more shells. How many shells does he wish to place in a row? and how many shells has he?

221. Richard proposes to divide a number of pears among his companions. If he gives them six apiece he will have six left, but to give them eight apiece he needs two more. How many companions has he? and how many pears does he propose to divide?

222. A father fifty years old has a son twenty-one years old. In how many years will the father's age be double that of the son?

223. How many years ago was the father's age thrice that of the son?

224. Emily spent one-third of her money for worsted, one-fourth for silk, one-sixth for linen thread, and had five cents left. How much money had she at first?

225. The sum of two-thirds and five-ninths of a certain number is six greater than the number itself. What is the number?

226. Two casks contained equal quantities of oil; but when ten gallons had been drawn from one cask into the other, the latter contained three times as much oil as the former. How much oil was there in each cask at first?

227. Said A to B, "I have only two-thirds as much money as you; but give me twelve dollars, and I shall then have twice as much as you." How much money had each?

228. In a certain orchard one-third of the trees bear apples, one-sixth pears, one-twelfth cherries, and the remaining twenty bear peaches. How many trees are there in the orchard?

229. What number is doubled by adding to it two-thirds of itself, one-fourth of itself, and five?

230. One piece of rope was three-fourths as long as another; but when twenty feet had been cut from the longer, and five feet from the shorter piece, the two were equal. How long were they at first?

40. Written Exercises: Problems.

231. How much rye at 45¢ a bushel must be mixed with fifty bushels of wheat at 60¢ a bushel, that the mixture may be worth 50¢ a bushel?

232. A debtor, having received a certain sum of money, proposes to divide it equally among his creditors. He finds, that, if he gives them $250 apiece, he will have $150 left; but, if he should try to give them $300 apiece, he would lack $500. How many creditors has he? and what is the sum he proposes to divide?

233. A peddler, being asked the price of some nutmegs, said that six of them cost as much more than a dime as eight of them cost less than a quarter of a dollar. Required the price of the nutmegs.

234. A market-woman having a basketful of eggs offered them at 18¢ a dozen; but, before she sold any, two dozen were broken by a dog upsetting her basket. Fortunately, however, she was able to sell the remainder at 21¢ a dozen, thereby getting as much as she would have got for the whole at the first price. How many dozen eggs had she in the basket?

235. The owners of a saw-mill, sharing its profits equally, make each 5 dollars a day. If the number of owners were three less, they would make 8 dollars each. Required the number of owners, and the total daily profits of the mill.

236. A shopkeeper, in consequence of bad book-keeping, knows neither the weight nor the first cost of a certain lot of tea he bought. He only remembers, that, if he had sold it at 30¢ a pound, he would have gained $1; but, if he had sold it at 25¢ a pound, he would have lost $1.50. What was the weight and first cost of the tea?

237. Two men, A and B, invested equal amounts of money in real estate. At the end of five years it was found that A's investment had increased by $4900, and B's lacked $1300 of having tripled, while the present value of both investments taken together was equal to five times the original value of either. What amount was invested by each?

238. A and B began to trade with equal amounts of capital. In the first year A tripled his capital, and $135 over; B doubled his capital, and $765 over. Now the amount of both their gains was five times the original capital of either. With what amount of capital did each begin?

239. In a mixture of alcohol and water, 5 gallons less than half the whole mixture was alcohol, and 15 gallons more than a third of it was water. How many gallons were there in the whole mixture?

240. Out of a cask of oil from which a third part had leaked away, 21 gallons were afterwards drawn, when the cask was found to be half full. How much did it hold when full?

241. A land-owner sold 125 acres more than one-fourth of his land, and there remain 15 acres less than a third part of it. How many acres had he at first?

242. A trader beginning with a certain capital doubled it during the first year, but took out $1500 to pay household expenses; using the remainder during the second year, he doubled it, and again took out $1500, when he found that he had $2\frac{1}{2}$ times as much capital as he had at the beginning. With how much capital did he begin?

243. Four children inherit property, sharing it in the following manner: the first takes $3000 less than half of the property, the second takes $1000 less than one-third of it, the third takes one-fourth of it, and the fourth takes $600 more than one-fifth of it. What was the whole amount of property? and what was the share of each child?

244. Find a number such that, if we add it to its half, the sum shall exceed 60 by as much as the number itself is less than 65.

245. At what rate of interest will $9000 yield the same income as $7500 at a rate 1 per cent higher?

246. A mother is 4 times as old as her son, but 3 years ago she was 7 times as old. What is the age of each now?

247. A's age is to B's as 7 to 8; but 6 years hence A's age will be to B's as 8 to 9. What are the present ages of each?

248. A man fifty years old has three sons, aged ten, twelve, and sixteen years respectively. In how many years will the united ages of the sons equal the age of the father?

249. A messenger started from this place 12 days ago; and to-day another is sent in pursuit, whose speed is to that of the first as 9 to 5. In how many days will the second overtake the first?

250. A and B start from different places to meet at a place half-way between. A starts at 7 o'clock A.M., and will arrive at 3 P.M. How much later may B start to arrive at the same hour, if he can walk sixteen miles while A walks fifteen miles?

251. Into one scale-pan is put eight-ninths as much sugar as into the other; but, if a quarter of a pound be transferred from one pan to the other, the two will balance. How many pounds of sugar are in each scale-pan?

252. A mast consists of two parts, the upper part being three-fifths as long as the lower. When put together, one part laps on the other 2 feet; and the height of the mast so made is 20 feet less than double the length of the lower part. What is the length of the lower part? of the whole mast?

253. There was a large fish caught whose head weighed 9 pounds; the tail weighed as much as the head and half the body; and the body weighed as much as the head and tail together. What was the weight of the fish?

Terms Known and Unknown in the Numerator of a Fraction

41. Oral Exercises.

254. What is one-half of the quantity $4x+6$?

255. What is one-third of the quantity $9+3x$?

256. What is two-thirds of the quantity $6x + 12$?
257. Divide the quantity $10x + 15$ by 5.
258. Divide the quantity $12x + 20$ by 4.

42. Let it be required to divide $2x + 5$ by 3. In this example, since the separate terms are not divisible by 3 without remainder, the quotient cannot be found at once in a simple form, as in the examples above. In such cases it is usually best to *express the division* by writing the dividend as the numerator, and the divisor as the denominator, of a fraction.

The division of $2x + 5$ by 3 is expressed by writing $\frac{2x+5}{3}$, which is read, "the quantity two x plus five divided by three;" or, "one-third of the quantity two x plus five."

Two-thirds of the same quantity would be found by doubling the numerator of the fraction above given, which would give $\frac{4x+10}{3}$.

259. What expresses one-fifth of the quantity $x + 1$?
260. What expresses one-eighth of the quantity $4x + 9$?
261. Express three-fifths of the quantity $x + 7$.
262. Express seven-eighths of the quantity $x - 1$.
263. Express five-sixths of $7 - 3x$.
264. Express three-fourths of $1 - x$.
265. If $\frac{x+1}{5} = 3$, what is the value of x?
266. If $\frac{x-3}{4} = 4$, what is the value of x?
267. What is the value of x in the equation $\frac{3x+6}{5} = \frac{2x}{3}$?
268. What is the value of x in the equation $\frac{2x-1}{3} + 2 = x$?
269. What is the value of x in the equation $\frac{5x}{6} = \frac{2x+8}{3}$?

Solve the equations:

270. $\frac{3-x}{5}=\frac{x}{10}$.

271. $\frac{2x+1}{3}=x-1$.

272. $\frac{2x}{3}+1=x-1$.

273. $\frac{8-x}{3}+5=6$.

274. $10+x=\frac{45+3x}{4}$.

275. $12-x=\frac{45-3x}{4}$.

276. $\frac{x+1}{4}=\frac{2x-4}{5}$.

277. $\frac{x+2}{3}=\frac{12-x}{4}$.

278. $\frac{8-x}{3}=\frac{x+1}{6}$.

279. $\frac{5-2x}{3}=\frac{2-x}{2}$.

43. Oral Exercises: Problems.

280. If a certain number be increased by one, and the sum divided by five, the quotient will be three. What is the number?

281. Said A to B, "I have two dollars more than you, and three-fifths of my money is equal to two-thirds of yours." How much money had each?

282. John is four years older than William, and five-sixths of William's age is equal to two-thirds of John's age. Required the age of each.

283. Mary is now ten years old, and Alice fifteen. How many years before Mary's age will be equal to three-fourths of Alice's age?

284. Equal quantities of wine were drawn from two casks, one containing twelve and the other fifteen gallons, when it was found that the quantity remaining in one cask was three-fourths that remaining in the other. How much was drawn from each?

285. A and B had equal amounts of money; but, when A had lent B four dollars, he had only two-thirds as much as B. How much had each at first?

286. A son's age is now three-fifths that of his father; but twelve years hence his age will be two-thirds that of his father. What are their ages now?

287. There are two boards, one twenty and the other twenty-eight feet long, and it is required to make the shorter two-thirds as long as the longer by cutting off a piece of the same length from each. How much must be cut off?

288. Walter has twice as many marbles as Henry; but, if he should give one to Henry, two-thirds of the number he would have left would be equal to the number Henry would have. How many marbles has each?

289. Separate the number thirty into two parts such that two-sevenths of the greater may be equal to two-thirds of the less.

290. Two travellers, A and B, set out from towns forty miles apart, and travelled toward each other. When they met, it appeared that two-fifths of the distance A had travelled was equal to two-thirds of the distance B had travelled. How far had each travelled?

291. If I multiply a certain number by three, subtract ten from the product, and divide the remainder by four, the result will be one-third the original number. What is the number?

292. Find a number from which, if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40.

44. **Written Exercises: Problems.**

293. Find a number such that if you multiply it by 5, subtract 20 from the product, divide the remainder by 7, and add 24 to the quotient, you obtain the number itself.

294. From a bin full of corn one horse consumed 8 bushels, and then another horse $\frac{3}{10}$ of what remained, when the bin was found to be $\frac{7}{12}$ full. How much did the bin hold?

295. Find three consecutive numbers such that the least is equal to four-ninths of the sum of the other two.

296. One side of a triangle is 2 inches longer, and another is 3 inches shorter, than the third side, while the shortest side of the triangle is equal to $\frac{3}{11}$ of the distance around it. Find the length of each side.

297. A privateer running 20 miles an hour gives chase to a schooner 18 miles off which is making way at the rate of 8 miles an hour. How many miles can the schooner run before she will be overtaken?

298. At what o'clock may it correctly be said that one-seventh of the time passed since noon is equal to one-ninth of the time to come before midnight?

299. A man holding a certain estate by lease for 99 years, and being asked how much of the time had already expired, answered that two-thirds of the time past was equal to four-fifths of the time to come. How much of the time had expired?

300. The length of a rectangular field exceeds its breadth by 8 rods, and its breadth is equal to $\frac{3}{14}$ of the distance around it. What is the area of the field?

301. A square grass-plat is surrounded by a walk five feet wide. The distance around the plat, measured on the inner edge of the walk, is eight-ninths of the distance around, measured on the outer edge of the walk. What is the length of one side of the plat?

302. Two inches from the middle point of a straight line is a point which divides the line into parts that are in the ratio of 5 to 6. How long is the line?

NOTE. Let $2x$ represent the length of the line.

303. A and B start at the same time from different towns to meet each other on the road, agreeing that the one who fails to reach the half-way point shall forfeit as many dollars as the other goes miles beyond that point to meet him. On meeting, A forfeits four dollars, for he can walk only five miles while B walks seven. What is the distance between the two towns?

304. A boatman rows twenty miles down a river and back again, in ten hours, the stream flowing uniformly all the time; and he finds he can row 3 miles against the current in the same time that he can row 5 miles with the current. Find how long it takes him to go down, how long to return, and the velocity of the stream.

Negative Terms subtracted.

45. Oral Exercises.

305. From 12 subtract 7.

306. From 12 subtract 1 *more* than 7, or $7 + 1$.

We first subtract 7, and have for a remainder $12 - 7$. We then subtract 1 more, and have for a final remainder $12 - 7 - 1$.

307. From 12 subtract 1 *less* than 7, or $7 - 1$.

We first subtract 7, and have for a remainder $12 - 7$. But this remainder is too small, for the subtrahend was to be, not 7, but 1 *less* than 7. We then add 1 to $12 - 7$, and have for the correct remainder $12 - 7 + 1$.

308. From 20 subtract $10 - 3$.

309. From 18 subtract $9 - 2$.

310. From 15 subtract $10 - 5$.

311. From $3x$ subtract x. From $3x$ subtract $x - 1$. Which remainder is the greater? How much? Why?

312. The minuend being $7x$, and the subtrahend $5x - 3$, what is the remainder?

313. From $8x + 5$ subtract $4x - 2$. Reduce the remainder to its simplest form by uniting terms.

In the following examples express the subtraction as above explained, and then unite terms:

314. From $9x + 3$ take $5x - 3$.

315. From $7x$ take $2x - 4$.

316. From 14 take $10 - x$.

317. From $8x$ take $3 - 2x$.

318. From $5x - 3$ take $9 - 4x$.

319. From $3x - 7$ take $2x + 7$.

It will be seen, that, in all these examples, the remainder is expressed by *writing the terms of the subtrahend with reversed signs after the terms of the minuend.* Thus, in examples 312 and 314, the subtrahend $5x - 3$ (which is the same as $+5x - 3$) becomes $-5x + 3$ in the remainder.

320. Multiply by 3 each member of the equation

$$4x - \frac{2x+5}{3} = 5.$$

In the first member, the minus sign shows that the fraction $\frac{2x+5}{3}$ is to be taken from $4x$. When this member has been multiplied by 3, the $4x$ becomes $12x$, and the fraction becomes $2x+5$; so that now $2x+5$ is to be taken from $12x$, which is expressed by writing $12x-2x-5$. The equation cleared of fractions, therefore, is

$$12x-2x-5=15.$$

321. Multiply by 2 each member of the equation

$$3x - \frac{x-1}{2} = 13.$$

In the first member the minus sign written after the $3x$ shows that the fraction $\frac{x-1}{2}$ is to be taken from $3x$; but, when the first member of the equation has been multiplied by 2, we shall have $x-1$ to be taken from $6x$, which is expressed by writing $6x-x+1$. The equation cleared of fractions, therefore, is

$$6x-x+1=26.$$

Clear of fractions and solve the following equations:

322. $x - \frac{x-1}{2} = 3.$

323. $x - \frac{x+1}{3} = 5.$

324. $2x + \frac{x-1}{2} = 7.$

325. $x - \frac{5-x}{2} = 2.$

326. $1 - \frac{1-x}{2} = 1.$

327. $\frac{x+1}{2} + \frac{x+3}{3} = 4.$

328. $\frac{x+2}{3} + \frac{x-2}{2} = 3.$

329. $\frac{x+1}{2} - \frac{x-1}{3} = 2.$

330. $\frac{x}{5} - \frac{12-x}{2} = 1.$

331. $\frac{9-x}{4} - \frac{3-x}{2} = 1.$

46. Oral Exercises: Problems.

332. Henry has one cent less than Robert. If the amount of Henry's money be subtracted from twice Robert's, the

remainder will be eight cents. How much money has each?

333. If from three times a certain number you subtract that number less five, the remainder will be twenty-one. What is the number?

334. Albert is four years younger than his brother; but, if twice his age be subtracted from thrice his brother's age, the remainder will be twenty-two years. How old is each?

335. Two boys started to walk to the city. When they had walked a mile, they found that half the remaining distance subtracted from the whole distance would leave a remainder of five miles. What was the whole distance to the city?

336. Alice has the same number of beads in each hand; but if she should change two from her left to her right hand, then one-third of the number in her right hand would exceed one-half the number in her left hand by one. How many beads has she in each hand?

337. From a brood of twelve chickens, a certain number has been killed by a cat. If one-fourth of the number remaining be subtracted from the number killed, the remainder will be two. How many chickens have been killed?

47. Written Exercises: Problems.

338. There is a number, from which if 5 be taken, the remainder divided by 4, and this quotient subtracted from 25, the result is equal to four-fifths of the original number. Find this number.

339. Separate the number 40 into two parts, such that three-fourths of their difference may equal one-half the smaller part.

340. Charles is now twelve years old, and Frank fifteen. In how many years will their ages be such, that half of Charles's age shall exceed a third of Frank's age by two years?

341. Mary had fifteen cents, and her sister eighteen. When each had bought a writing-book, it was found that one-third of the money Mary had left subtracted from one-half the money her sister had left, would leave a remainder of three cents. What was the price of the writing-book?

342. Two workmen were employed for fifty days, at one dollar a day each. A spent 12 cents a day less than B did, and at the end of the fifty days found that he had saved twice as much as B, and one dollar more. What did each spend a day?

343. Three consecutive numbers are such, that half of the least, a third of the next, and a fourth of the greatest added together amount to the greatest. What are the numbers?

344. A fund of $40000 is invested partly in 4-per-cent bonds, and partly in $4\frac{1}{2}$-per-cent bonds. The whole income is $1725. How much is invested in each kind of bonds?

345. Separate 80 into two such parts that one-tenth of the smaller part taken from one-fifth of the greater shall leave 10 for a remainder.

346. A had twice as much money as B. When A had gained $30 and B lost $40, A gave to B $\frac{3}{10}$ as much as B had left, whereupon it appeared that A had what he had in the beginning, and 20 per cent more. How much had each in the beginning?

347. A grocer bought two casks full of oil, one of which held twice as much as the other. From each cask he sold ten gallons, and then drew from the larger one-third of the remaining oil and poured it into the smaller, and found that the two casks now contained equal quantities of oil. How much did each cask hold when full?

348. Two travellers, A and B, took equal sums of money for their journey. When A had spent $15 and B $25, it appeared that the fifth part of what A had left exceeded $20 by as much as the fifth part of what B had left fell short of $20. What did each have at the outset?

48. Miscellaneous Problems.

349. A person bought a carriage, horse, and harness for $675; the horse cost twice as much as the harness, and the carriage cost twice as much as the horse and harness together. What did he pay for each?

350. There are in a purse 14 coins which are worth a dollar. Some of them are dimes, and the rest are nickel 5-cent pieces. How many of each are there?

351. A drover bought some cows at $45 apiece, and an equal number of sheep at $7 apiece, paying for all $468. What number of each did he buy?

352. A box of tea worth $6 was made by mixing equal quantities of three sorts: one sort worth 60¢ a pound, another 48¢, and another 42¢. How many pounds were there in the box?

353. If I buy a certain number of pounds of beef at 25 cents a pound, I shall have 25 cents left; but, if I buy the same number of pounds of mutton at 15 cents a pound, I shall have $1.25 left. How much money have I?

354. A grocer has two sorts of sugar, one worth 9 cents and the other 13 cents a pound, of which he wishes to make a mixture worth 12 cents a pound. How many pounds of each sort must he take to make a mixture of 100 pounds?

355. A paymaster, wishing to draw for use on pay-day the sum of $11360, requested the teller to make up the sum with bank-bills of different denominations, as follows: a certain number of 100's, twice as many 50's as 100's, twice as many 20's as 50's, twice as many 10's as 20's, twice as many 5's as 10's, twice as many 2's as 5's, and twice as many 1's as 2's. How many bills of each denomination were needed?

356. Two persons, A and B, invest equal sums of money in trade. A gains $12000 and B loses $4000, and B's money is now only one-half of A's. How much did each invest?

357. Separate 20 into two parts such that one part shall be 20 times as great as the other.

358. Separate the number 104 into two parts such that, when one is divided by 7 and the other by 3, the sum of the quotients may be 28.

359. Separate the number 90 into two such parts that half of the greater added to double the smaller shall give for a sum the original number, 90.

360. There are two places 154 miles apart, from which two persons, A and B, set out at the same time, intending to meet on the road. A walks 3 miles every 2 hours, and B 5 miles every 4 hours. How long and how far will each walk before they meet?

361. A boy is sent on an errand to a town 8 miles off, and walks at the rate of 4 miles an hour: ten minutes later another boy is sent to fetch him back, walking $4\frac{1}{2}$ miles an hour. How far from the town will the latter overtake the former?

362. At 9 o'clock this morning a courier passed through the village, riding at the rate of $31\frac{1}{2}$ miles in 5 hours; and at 5 o'clock this afternoon, another courier passes in pursuit at the rate of $22\frac{1}{2}$ miles in 3 hours. When will the second overtake the first, supposing that they travel continuously?

363. A woman possesses two silver cups, and only one cover for both. The first cup weighs 12 ounces, and, with the cover put on, it weighs twice as much as the other cup; but, when the other cup has the cover, it weighs three times as much as the first. Required the weight of the second cup, and that of the cover.

364. Three masons, A, B, and C, can build a wall in six days. A works half as fast, and B two-thirds as fast, as C. In how many days would each build the wall working alone?

365. A and B find a bag of silver dollars. A takes out 2 dollars and $\frac{1}{6}$ of what remain, then B takes out 3 dollars and $\frac{1}{6}$ of what then remain, and they find that each has taken out the same number of dollars. How many dollars were there in the bag?

366. Find two consecutive numbers such that a half and

a fifth part of the first taken together shall be equal to a third and a fourth part of the second taken together.

367. A boatman who can row 6 miles an hour in still water rows a certain distance down stream and back again in 3 hours. How far did he row, supposing the stream to flow at the rate of 3 miles in 2 hours?

368. The straight line AB is divided at the point M, so that the part AM is to the part MB as 5 to 9, and at the point N, so that the part AN is to the part NB as 9 to 5. If the distance from M to N is 18 inches, what is the length of AB?

Solve the following equations:

369. $\frac{2x+1}{2}=\frac{7x-1}{6}$.

370. $\frac{x-4}{3}-\frac{x-3}{4}=\frac{x-2}{5}-\frac{x-1}{6}+\frac{1}{4}$.

371. $\frac{3x-3}{4}+\frac{7x-5}{10}=\frac{4x-2}{3}$.

372. $\frac{x+3}{4}-\frac{x-1}{2}=\frac{x-2}{3}-\frac{x+5}{6}-1$.

373. $0.2x-0.325=0.15x$.

374. $x+0.06x=52470$.

375. $x-0.045x=10000$.

376. $x+0.18x=5000$.

377. $x-0.0205x=400$.

378. $x+0.07x=3210$.

379. $2x+2.5x=900$.

380. $\frac{7x+9}{8}-\frac{3x+1}{7}=\frac{9x-13}{4}-\frac{249-9x}{14}$.

381. $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$.

382. $\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}=23\frac{1}{3}$.

SECTION III.

OPERATIONS ON ALGEBRAIC QUANTITIES.

49. Algebraic expressions, as we have seen, are formed by combining symbols so as to express one or more of the four arithmetical operations of addition, subtraction, multiplication, and division. The exercises in the last section afford illustrations of the various ways in which numbers may be expressed by combining algebraic symbols. We shall now inquire whether every combination of symbols expresses a number.

50. Oral Exercises.

1. What is the value of $a + b$, when $a = 7$ and $b = 4$?

2. Can the value of $a + b$ always be found when the values of a and b are separately given?

3. What is the value of ab, when $a = 5$ and $b = 6$?

4. Can the value of ab always be found when the values of a and b are separately given?

5. What is the value of $\frac{a}{b}$, when $a = 12$ and $b = 3$? When $a = 13$ and $b = 4$? When $a = 2$ and $b = 5$?

6. Can the value of $\frac{a}{b}$ always be found when the values of a and b are separately given?

7. What is the value of $a - b$, when $a = 7$ and $b = 3$?

Since the sum, product, and quotient of two numbers can always be found by the ordinary operations of arithmetic, the expressions $a + b$, ab, and $\frac{a}{b}$ always express numbers, and are said to be perfectly general. The remainder, expressed by $a - b$, however, presents a different case, which requires further consideration.

Positive and Negative Numbers.

51. What is the value of $a - b$, when $a = 7$ and $b = 10$?

When the value of b is greater than that of a, the expression $a - b$, taken in an arithmetical sense, represents an impossible subtraction; for it is evident that no more can be taken away from a than a contains. In an algebraic sense, however, such a subtraction is regarded as possible.

The remainder obtained by subtracting a greater number from a less is expressed by writing the arithmetical difference between the two numbers with the minus sign.

Thus, when 10 is subtracted from 7, the remainder is -3; when 9 is subtracted from 5, the remainder is -4; and, when 2 is subtracted from 1, the remainder is -1. These remainders, -3, -4, and -1, are called ***negative numbers.***

52. Negative numbers, then, are used to give a meaning to such expressions as $a - b$, in the case where b is greater than a. In view of this circumstance, we may define two classes of numbers as follows:

Positive numbers are such as may be obtained by subtracting a less number from a greater.

Negative numbers are such as may be obtained by subtracting a greater number from a less.

Positive numbers are written with the sign plus; negative numbers, with the sign minus. When not needed to mark this distinction, the sign plus is often omitted: the sign minus is never omitted.

53. Numbers regarded as positive or negative are, for the sake of distinction, called ***algebraic numbers.*** Arithmetical numbers, on the other hand, are not subject to this distinction.

54. The series of integral numbers used in arithmetic is,

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\ \ldots$$

indefinitely extended one way, but limited the other.

The series of integral numbers used in algebra is,

$$\ldots -6, -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, +6 \ldots$$

indefinitely extended both ways.

It will assist the beginner if he will keep this series of algebraic numbers before his mind. He may think of addition as *a counting forward* (that is, toward the right) in this series from a given number as many places as there are units in the number to be added. Thus 3 is added to 2 by beginning at $+2$, and counting three places forward, which gives $+5$. Again, 6 is added to -4 by beginning at -4, and counting six places forward, which gives $+2$. Subtraction may be thought of as *a counting backward* in this series from a given number as many places as there are units in the subtrahend. Thus 4 is subtracted from 6 by counting four places backward from $+6$, which gives $+2$. Again, 8 is subtracted from 3 by counting eight places backward from $+3$, which gives -5.

55. The senses in which the words *greater* and *less* are to be understood in algebra may be fixed by reference to the series of algebraic numbers above given.

Every number in that series is deemed to be greater than any preceding and less than any following number.

Thus $+2$ is greater than $+1$, and less than $+3$; but -2 is less than -1, and greater than -3.

In this sense it is said that any positive number is *greater than zero*, and any negative number is *less than zero*.

NOTE. The words *greater* and *less*, when used in an arithmetical sense, refer to the absolute values of numbers, no account being taken of their character as positive or negative. Thus -5 is *arithmetically greater* than $+3$, and -5 is *arithmetically equal* to $+5$. Indeed, it is frequently convenient to speak of two numbers as being *equal, but of opposite signs.* The meaning is that they are arithmetically equal; that is, equal, if signs are disregarded.

In this book the words *greater* and *less* are used in the algebraic sense unless the contrary is expressed or implied.

Use and Interpretation of Positive and Negative Numbers.

56. ILLUSTRATIVE PROBLEMS. I. A man bought a horse for $150, and sold it for $170. How much did he gain?

II. A man bought a horse for $150, and sold it for $130. How much did he gain?

Taking no account of the particular amounts of money mentioned in these two similar problems, we assume that both may be solved by the same rule, which is this: *To find the gain, subtract the cost from the price received.*

In I we find the gain to be 170 − 150 dollars, or + 20 dollars.

In II we find the gain to be 130 − 150 dollars, or − 20 dollars.

Comparing these results with the statements in the two problems, we learn that a gain of + 20 dollars is to be interpreted as a *real gain*, but a gain of − 20 dollars is to be interpreted as the reverse of a gain; that is, as a *loss*.

III. A man having in his possession property worth $6000 owes $2000. What is the net value of his estate?

IV. A man having in his possession property worth $6000 owes $10000. What is the net value of his estate?

Assuming the same rule for these two similar problems, we find the net value of the estate to be,

in III, 6000 − 2000, or + 4000 dollars;
and in IV, 6000 − 10000, or − 4000 dollars.

These two examples show that, when + 4000 dollars is interpreted as actual *wealth*, − 4000 dollars is to be interpreted as actual *indebtedness.*

57. We have seen that, when a man's profits are reckoned in positive numbers, his losses are to be reckoned in negative numbers; and when his possessions are reckoned in positive, his debts are to be reckoned in negative numbers. So it is with his income and expenses, his purchases and sales, or items debited and items credited in any book-account. Again, distances east may be reckoned in positive numbers, and distances west in negative numbers; and so with distances north and south, forward and backward, right and left, up and down, or any direction and the opposite.

North latitude is often reckoned in positive, and south latitude in negative, numbers; so of longitude east and west; so of time reckoned forward or back from a given date; so of pressure and resistance, weight and lifting force, or any two forces acting in opposite directions; *and so, indeed, of any quantities whatsoever which are opposed to one another in kind or in effect.*

58. The statements above made may be summed up in the following

Rule for the Interpretation of Signs.

Having determined the meaning and effect of the sign plus, give the opposite meaning and effect to the sign minus.

59. Exercises.

8. A merchant's gross income during the year was $16000, and his expenses $12000. What was his net income?

9. What was his net income, if his gross income was $16000, and his expenses $17000?

10. A man having property in his possession worth $8000 owes $3000. What is the net value of his estate?

11. A man having property in his possession worth $8000 is indebted to the amount of $13000. What is the net value of his estate?

12. A trader beginning the year with $2000 made during the year $1000 profit and $500 loss. How much had he at the end of the year?

13. A merchant was engaged in business three years. During the first year he lost $500; during the second, he gained $1500; and, during the third, he lost $1200. How much profit did he make on the three years' business?

14. How much does a clerk save, if his salary is $800 a year and his expenses $80 a month?

15. A man bought a horse for $150, and sold him for $170. How much did he lose?

16. A person walks forward 10 paces, and then backward 7 paces. How far is he in advance of his starting-point?

17. A person walks forward 10 paces, and then backward 15 paces. How far is he in advance of his starting-point?

18. A ship leaving latitude 42° north sails southerly through 12° of latitude. What latitude does she reach?

19. A ship leaving latitude 25° north sails southerly through 40° of latitude. What latitude does she reach?

20. Two boys can row their boat 5 miles an hour in still water. How fast can they row up stream against a current that flows 6 miles an hour?

21. At sundown the mercury in a thermometer stands at $+10$ degrees, but during the night it falls 15 degrees. How high does it stand in the morning?

22. At sunrise on a cold day the mercury in a thermometer stood at -17 degrees, but during the forenoon it rose 25 degrees. How high did it stand at noon?

23. What is the sum of -17 and $+25$? Of -15 and $+20$? Of $+20$, $+20$, and -50?

Note. It will be well to draw diagrams to illustrate the three following exercises.

24. A ship sails on a course that takes her 20 miles to the east, and 10 miles to the north, of her starting-point; then on a course that takes her 20 miles farther to the east, and 40 miles to the south; and then on a course that takes her 50 miles to the west, and 50 miles farther to the south. How far east of her starting-point is she now? How far north?

A land-surveyor measuring the sides of a field calculates the distance north or south, and the distance east or west, that he goes in passing from corner to corner along each side of the field. Distances north and south are reckoned as positive and negative respectively, and are called *differences of latitude.* Distances east and west are reckoned as positive and negative respectively, and are called *departures.* The sum of the differences of latitude is the distance north, and the sum of the departures is the distance east, from the starting-point to the stopping-point. If he goes clear around the field, these sums will each be zero.

25. Find how far north and east from his starting-point a surveyor has gone in measuring three sides of a field of which the differences of latitude and departures are as follows:

SIDE.	DIFF. OF LATITUDE.	DEPARTURE.
First,	$+30$ chains,	$+20$ chains.
Second,	$+10$ chains,	-50 chains.
Third,	-25 chains,	$+10$ chains.

26. Find how far north and east of his starting-point a surveyor is after measuring the sides of a field which give the following results:

SIDE.	DIFF. OF LATITUDE.	DEPARTURE.
First,	$+10$ chains,	$+30$ chains.
Second,	-20 chains,	$+15$ chains.
Third,	$+40$ chains,	$+15$ chains.
Fourth,	$+20$ chains,	-20 chains.
Fifth,	-50 chains,	-40 chains.

27. Mary is 14 years old, and her sister 8 years. In how many years will Mary be twice as old as her sister?

Letting x represent the number of years hence, the equation may be formed and solved in the usual way, when it will be found that the value of x is a negative number. Mary *will be* twice as old as her sister a *negative number* of years *hence.* This is the same as saying that she *was* twice as old a *positive number* of years *ago.*

In a similar way interpret negative values of x in the following problems.

28. A father's age is 56 years, and that of his son 16 years. In how many years hence will his age be equal to five times that of his son?

29. What number is that whose third part is greater than its half by 4?

30. There are two purses, one containing $7, and the other $15. It is required to make the latter contain three times as much as the former, by putting into each the same additional number of dollars. How many dollars must be put into each?

31. A rectangular field is 16 rods long and 12 rods wide. By how much should the width be increased, that the field may contain 1 acre?

32. B's wealth is $3000 greater than A's; but, if twice A's wealth be increased by adding $8000, the sum will be equal to three times B's wealth. Find the wealth of each.

33. Says A to B, "My wealth is now double yours; but, if each of us should acquire $6000, your wealth would then be double mine." Find the wealth of each.

Operations on Algebraic Numbers.

60. Algebraic addition, subtraction, multiplication, and division, differ from the arithmetical operations of the same names in the circumstance that, in algebra, account is taken of the positive or negative character of the numbers operated on, and of the positive or negative character of the results. The following exercises will illustrate the subject.

Addition.

61. ILLUSTRATIVE EXAMPLES. I. To $+4$ add $+3$.

Counting forward three places from $+4$, we find $+7$ to be the sum of $+4$ and $+3$. In this case the algebraic sum is the arithmetical sum regarded as positive.

II. To -5 add $+8$.

Beginning at -5, and counting eight places forward, we find $+3$ to be the sum of -5 and $+8$. In this case the algebraic sum is the arithmetical difference regarded as positive.

III. To -5 add $+3$.

Counting three places forward from -5, we find -2 to be the sum of $+3$ and -5. In this case the algebraic sum is the arithmetical difference regarded as negative.

IV. To $+4$ add -3.

Since the effect of adding a positive number is to produce a sum *greater* than the number added to, the effect of adding a negative number must be to produce a sum *less* than the number added to. (See Art. 58.) Counting *backward*, then, three places from $+4$, we find $+1$ to be the sum of $+4$ and -3. In this case the algebraic sum is the arithmetical difference regarded as positive.

V. To $+4$ add -6. *Ans.* -2.

VI. To -3 add -2. *Ans.* -5.

62. From the above examples may be derived a

Rule for the Addition of Algebraic Numbers.

When two numbers have like signs, find their arithmetical sum, and give it the sign of the numbers added; when two numbers have unlike signs, find their arithmetical difference, and give it the sign of the arithmetically larger of the numbers added.

63. Oral Exercises.

	(34.)	(35.)	(36.)	(37.)	(38.)	(39.)	(40.)	(41.)
To	-3	-5	$+7$	-6	$+5$	$+1$	$+4$	-5
add	$+4$	$+3$	-4	-2	$+2$	-4	-4	$+5$

42. A man possessing \$4000 receives a gift of \$3000. How much property has he then?

43. A man being \$500 in debt receives a gift of \$800. What is his wealth then?

44. A man being \$500 in debt receives a gift of \$300. What is his wealth then?

45. A man having property to the amount of \$4000 adds to it a debt of \$3000. What is his wealth then?

46. Add a debt of $600 to the wealth of a man who has $400 worth of property. What will his wealth be then?

47. Add a debt of $200 to the wealth of a man who already owes $300 more than he can pay. What will his wealth be then?

Add the following:

(48.)	(49.)	(50.)	(51.)	(52.)
$+1$	$+3$	-7	$+8$	$+20$
-3	$+2$	$+9$	-1	$+60$
$+2$	-4	-8	-6	$+40$
-4	-3	$+6$	-5	-30
$+3$	$+1$	-1	$+2$	-50
-2	-4	-5	$+9$	-70
-3	$+6$	$+2$	$+6$	-20
$+3$	$+5$	-4	-1	$+80$
$+4$	-7	$+9$	$+3$	$+30$

Subtraction.

64. As subtraction is the reverse of addition, we must, in subtracting a *positive* number, count *backward*, and, in subtracting a *negative* number, count *forward*, in the series of algebraic numbers.

This is illustrated by the following examples, which may be solved by counting in the series of algebraic numbers backward or forward, as the case may be. The results may be compared with the numbers given in the corresponding examples of Art. 61.

ILLUSTRATIVE EXAMPLES.

I. From $+7$ subtract $+3$. *Ans.* $+4$.
II. From $+3$ subtract $+8$. *Ans.* -5.
III. From -2 subtract $+3$. *Ans.* -5.
IV. From $+1$ subtract -3. *Ans.* $+4$.
V. From -2 subtract -6. *Ans.* $+4$.
VI. From -5 subtract -2. *Ans.* -3.

65. From these examples it is evident that *the subtraction of a positive number is equivalent to the addition of an equal negative number, and that the subtraction of a negative number is equivalent to the addition of an equal positive number.*

Hence a

Rule for the Subtraction of Algebraic Numbers.

Reverse the sign of the subtrahend, and then add it to the minuend.

66. Oral Exercises.

53. From $+4$ subtract -7. To $+4$ add $+7$. Are the results the same? Why?

	(54.)	(55.)	(56.)	(57.)	(58.)	(59.)	(60.)	(61.)
From	-4	$+5$	-4	-4	$+5$	$+5$	$+5$	-3
subtract	$+3$	-3	-2	-6	$+3$	$+8$	$+5$	-3

62. A man having property worth $7000 gives away $3000. What is his wealth then?

63. A man taking account of his possessions and his debts, found that his wealth amounted to $1000. From this was afterwards removed a debt of $3000. What was his wealth then?

64. Subtract the reading of the thermometer at sunrise, $-12°$, from its reading at noon, $+25°$, to find how much the mercury has risen during the forenoon.

65. Subtract the reading of the thermometer at noon, $+25°$, from its reading at sundown, $+5°$, to find how much the mercury has risen during the afternoon.

66. Subtract the latitude a ship leaves, $-25°$, from the latitude she arrives at, $+40°$, to find how many degrees of latitude she has sailed through, and which way she has sailed.

67. Subtract latitude $+30°$ from latitude $-10°$. Which way does a ship sail to go from the former to the latter? How does the result indicate this?

Multiplication.

67. ILLUSTRATIVE EXAMPLES. I. Multiply $+4$ by $+3$.

This is to unite three $+4$'s. But $+4+4+4=+12$.
Hence $(+4)\times(+3)=+12$.

II. Multiply -4 by $+3$.

This is to unite three -4's. But $-4-4-4=-12$.
Hence $(-4)\times(+3)=-12$.

III. Multiply $+4$ by -3.
IV. Multiply -4 by -3.

Here we must determine the effect of using -3 as a multiplier. According to Art. 58 this effect must be the opposite of that produced by $+3$ when used as a multiplier. Now, when the multiplier is positive, as in Exs. I. and II. above, the product is like the multiplicand, being positive when the latter is positive, and negative when it is negative. This accords with a principle of arithmetic that the product is of the same denomination as the multiplicand. The effect of using a negative multiplier should be to reverse this, and make the product negative when the multiplicand is positive, and positive when the multiplicand is negative.

Hence $(+4)\times(-3)=-12$, and $(-4)\times(-3)=+12$.

In the above examples, what is the sign of the product when the signs of the multiplicand and multiplier are both plus or both minus? What is the sign of the product when the signs of the multiplicand and multiplier are unlike?

68. From these examples we see that the meanings to be assigned to positive and negative multipliers are the following:

A *positive* multiplier signifies that the multiplicand *itself* is to be taken as many times as there are units in the multiplier.

A *negative* multiplier signifies that the multiplicand *with its sign reversed*, is to be taken as many times as there are units in the multiplier.

Hence a

Rule for the Multiplication of Algebraic Numbers.

Find the arithmetical product of the numbers, and, if their signs are alike, give the product the sign plus, if unlike, give it the sign minus.

69. Oral Exercises.

68. Multiply $+7$ by -5.
69. Multiply -6 by $+4$.
70. Multiply $+9$ by $+6$.
71. Multiply -12 by -12.
72. Multiply $+1$ by $+1$.
73. Multiply -1 by $+1$.
74. Multiply $+1$ by -1.
75. Multiply -1 by -1.

Division.

70. ILLUSTRATIVE EXAMPLES. I. Divide $+12$ by $+3$.

We seek a number which multiplied by $+3$ will give $+12$ as a product. By Art. 67, Ex. I., this number is $+4$.

Hence $(+12) \div (+3) = +4$.

II. Divide -12 by $+3$.

By Art. 67, Ex. II., the quotient is -4.

Hence $(-12) \div (+3) = -4$.

III. Divide $+12$ by -3.

By Art. 67, Ex. IV., the quotient is -4.

Hence $(+12) \div (-3) = -4$.

IV. Divide -12 by -3.

By Art. 67, Ex. III., the quotient is $+4$.

Hence $(-12) \div (-3) = +4$.

In these examples, what is the sign of the quotient when the signs of the dividend and divisor are alike? What is the sign of the quotient when the signs of the dividend and divisor are unlike?

71. From these examples may be derived a

Rule for the Division of Algebraic Numbers.

Find the arithmetical quotient of the numbers, and, if their signs are alike, give the quotient the sign plus, if unlike, give it the sign minus.

72. Oral Exercises.

76. Divide -36 by $+9$.
77. Divide -25 by -5.
78. Divide $+63$ by -7.
79. Divide $+48$ by $+8$.
80. Divide $+1$ by $+1$.
81. Divide -1 by $+1$.
82. Divide $+1$ by -1.
83. Divide -1 by -1.

73. From Arts. 68 and 71 may be derived a

Brief Rule for Signs in Multiplication and Division.

Like signs give PLUS; *unlike signs give* MINUS.

Monomials.

74. An algebraic quantity expressed by one symbol or by symbols so combined as to express only multiplication or division is a ***monomial,*** or quantity of *one term.*

Thus x, $3x$, ay, $5mz$, and $-7abc$ are monomials.

75. Monomials, like algebraic numbers, may be either positive or negative, and are written with the signs plus or minus accordingly.

Thus $+5ax$ is a positive, and $-5ax$ a negative monomial.

76. Quantities multiplied together are called ***factors*** (makers) of the product; but, with reference to each other, they are called ***co-efficients*** (helpers in making).

Thus in the quantity $5ax$, which is composed of the factors 5, a, and x, the co-efficient of ax is 5, and the co-efficient of x is $5a$.

77. Factors expressed by figures are called ***numerical factors;*** expressed by letters, ***literal factors.*** The same distinction is observed with regard to numerical and literal co-efficients; but the word co-efficient used alone commonly means the numerical factor or factors of a monomial.

Thus, the co-efficient of $3my$ is 3, and the co-efficient of $4 \times 5abxy$ is 4×5.

When no numerical co-efficient is written, the co-efficient 1 is always understood. Thus ax and $1ax$ mean the same thing; and so do $-ax$ and $-1ax$.

78. A product of equal factors is called a ***power.***

Thus aa is the second power of a,
aaa is the third power of a,
$aaaa$ is the fourth power of a,
and so on.

79. When a quantity is used more than once as a factor, its symbol is commonly written but once, and the number of times it is used as a factor is indicated by a small figure placed a little above and to the right of the symbol. The number so written is the ***exponent.***

Thus, instead of aa, it is usual to write a^2; instead of aaa, a^3; and so on.

When no exponent is written, the exponent 1 is always understood. Thus a and a^1 mean the same thing.

The expressions a^2 and a^3 are for brevity often read "a square" and "a cube," instead of "a to the second power" and "a to the third power."

If the quantity whose power is to be indicated is expressed, not by a single letter, but by a combination of letters, then the whole expression is enclosed in brackets, and the exponent is written outside the brackets, a little above, and to the right. Thus the second power of $3abc$ is indicated by writing $(3abc)^2$; the third power of $-4x$, by writing $(-4x)^3$; the second power of a^2, by writing $(a^2)^2$; and the fourth power of $a+x$, by writing $(a+x)^4$.

The exponent is often represented by a letter. Thus, if a is used m times as a factor, the power is denoted by a^m, and the expression is read, "the m^{th} power of a," or "a to the m^{th} power."

Multiplication.

80. Illustrative Examples. I. Multiply $3ab$ by $4xy$.

$$\begin{aligned}(3ab) \times (4xy) &= 3ab4xy\\ &= 3 \times 4abxy,\\ &= 12abxy.\end{aligned}$$

Since a product contains all the factors of the multiplicand and of the multiplier, but no others, we may express the product in this example by writing the factors of $3ab$ and of $4xy$ together so as to express one product; thus, $3ab4xy$. Again, since the order of factors may be changed without changing the value of the product, we may write the numerical factors first; thus, $3 \times 4abxy$. Putting 12 in place of 3×4, we have the final result $12abxy$.

II. Multiply $4m$ by $-7x$.

Since monomials are numbers, the rule for signs already explained in the case of positive and negative numbers (Art. 73) applies to monomials as well.

Hence $(+4m) \times (-7x) = -28mx$.

III. Multiply a^3 by a^4.

As there are three factors, each equal to a, in the multiplicand, and four more in the multiplier, the product must contain seven factors each equal to a.

$$a^3 \times a^4 = aaa \times aaaa$$
$$= aaaaaaa$$
$$= a^7.$$

In this example the product a^7 has an exponent equal to the sum of the exponents of the two factors a^3 and a^4; that is, $a^3 \times a^4 = a^{3+4} = a^7$. Moreover, it is plain that the same would be true in all similar examples. Thus, $a^2 \times a^3 = a^{2+3} = a^5$; $a^3 \times a = a^3 \times a^1 = a^{3+1} = a^4$; and, in general, if a^m and a^n represent any two powers of a, their product is represented by a^{m+n}; that is, $a^m \times a^n = a^{m+n}$.

An equation which is used to express a general rule or statement is called a ***formula.*** Thus the general statement that *the product of two powers of the same quantity has an exponent equal to the sum of the exponents of the factors* is expressed by the equation

$$a^m \times a^n = a^{m+n}$$

and this is known as *the formula for the multiplication of powers.*

IV. Multiply $-4a^2bx$ by $5ab^3x^2y$.

This example merely combines points explained in the last three.

$$(-4a^2bx) \times (5ab^3x^2y) = -4 \times 5a^2abb^3xx^2y = -20a^3b^4x^3y.$$

V. Find the continued product of $-3a^2b$, $-5ab^2c$, $-3cy$, and $2ax^2$.

The product of the first two quantities will, by the rule, be positive; this multiplied by the third will give a negative result; and this multiplied by the fourth will give a negative result: hence the product sought will have the minus-sign. This being settled, we have only to find the arithmetical product of the co-efficients, and add the exponents of the like literal factors.

$$(-3a^2b)\times(-5ab^2c)\times(-3cy)\times(2ax^2)$$
$$=-(3\times5\times3\times2)\,a^2aabb^2ccx^2y$$
$$=-90a^4b^3c^2x^2y.$$

It will be noticed in this example that an odd number (three) of the given monomials are negative, and the product is negative. If an even number (two or four) of them had been negative, the product would have been positive.

81. From these examples may be derived a

Rule for the Multiplication of Monomials.

For the co-efficient of the product take the arithmetical product of the co-efficients of the given monomials.

With this co-efficient write all the different literal factors, each with an exponent equal to the sum of its exponents, in the several given monomials.

Make the sign of the product minus when an odd number of the given monomials are negative; otherwise make it plus.

82. Exercises.

84. Multiply ab by $7xy$.
85. Multiply $3abc$ by $9bcm$.
86. Multiply $-ab$ by $3c$.
87. Multiply $-15m^2$ by $5n^2$.
88. Multiply $-m$ by n.
89. Multiply m by $-n$.
90. Multiply $-m$ by $-n$.
91. Multiply a by $-a$.
92. Multiply $-4a$ by -1.
93. Multiply $-3ab$ by $-3a$.
94. Multiply $3a$ by a^3.
95. Multiply $-5m^3$ by $-7m^2n^2$.

96. Multiply $-a^3$ by $-a$.
97. Multiply m^4 by n^3.
98. Multiply $4m$ by $3n$.
99. Multiply together abc, bcd, cde, dea, and eab.
100. Multiply together $-xy$, yz, $-zw$, and wx.
101. Multiply $4ax$ by $4ax$, and the product by $4ax$; that is, find the third power of $4ax$.
102. Find the fourth power of $2my$.
103. Find the value of $(6xy)^3$.
104. Multiply $-a$ by $-a$.
105. Multiply $-m$ by $-m$, and the product by $-m$
106. Find the value of $(-a)^4$.
107. Find the value of $(-a)^5$.
108. Find the second and fourth powers of -1.
109. Find the third and fifth powers of -1.
110. What is the value of any odd power of -1?
111. What is the value of any even power of -1?
112. What is the value of any power of $+1$?
113. Multiply $(3ab)^4$ by $(2abc)^3$.
114. Multiply $(-2xy)^4$ by $(-2am)^3$.
115. Find the value of $(a^2)^4$.
116. Find the value of $(2ax^2y^3)^2$.
117. Multiply $(-am^2)^3$ by $(2x^2y)^2$.
118. Multiply together $3k^2$, $4k^3$, and $5k^4$.
119. Multiply together n^x, n^y, and n^z.
120. Multiply a^m by a^2.
121. Multiply a^m by a^m.
122. Multiply $-5x^p$ by $8x^q$.
123. Multiply a^{m-n} by a^n.
124. Multiply a^{n-1} by a.
125. Multiply x^{p-1} by x^2.

Reduce each of the following expressions to its simplest form by performing the operations indicated.

126. $(15ab^2c) \times (6am^2)$.
127. $(3mn) \times (-am) \times (5bn)$.
128. $(-1) \times (-2) \times (-3) \times (-4)$.

129. $(-m)^3 \times (-2m)^2 \times (-3m)$.
130. $(ab)^3 \times (2a)^2$.
131. $(-amx) \times (3by^2)^4$.
132. $(-h^2k)^3 \times (-l^2) \times (-3k)^2$.
133. $(-a^3) \times (-a^2) \times (-a)$.
134. $(-a)^3 \times (-a)^2 \times (-a)$.
135. $(12a^2y)^3$.
136. $(-16l^3m^2)^2$.
137. $(x^p)^2$.
138. $(a^mb^n)^3$.
139. $(m^rn^s)^q$.

Division.

83. ILLUSTRATIVE EXAMPLES. I. Divide $6ab$ by $2a$.

We seek a quantity which multiplied by $2a$ will give $6ab$ as a product. This quantity will be composed of all the factors of $6ab$ which are not in $2a$. These are 3 and b, and the quantity sought is $3b$.

Hence $6ab \div 2a = 3b$.

It will be seen that we removed from the dividend the factors of the divisor, and the factors of the quotient were left.

II. Divide a^5 by a^2.

Removing from a product of five equal factors two of them will leave three of them.

Hence $a^5 \div a^2 = a^3$.

In this example the quotient a^3 has an exponent equal to that of the dividend diminished by that of the divisor: that is,

$$a^5 \div a^2 = a^{5-2} = a^3.$$

The same would be true in all similar cases.

To express this in a general form we write the equation

$$a^m \div a^n = a^{m-n}$$

which is *the formula for the division of one power by another power of the same quantity.*

III. Divide a^5 by a^5.

When the divisor is equal to the dividend, the quotient is 1. Hence $a^5 \div a^5 = 1$.

If we subtract exponents, as in the last example, we find that $a^5 \div a^5 = a^{5-5} = a^0$.

By Axiom 7, the two results here obtained are equal; that is, $a^0 = 1$.

The meaning of 0 when used as an exponent is defined by the equation $a^0 = 1$. *Any quantity with zero for an exponent is equal to 1.*

IV. Divide $12\,a^4 b^2 c^3$ by $2\,a\,b^2 c^2$.

$$12\,a^4 b^2 c^3 \div 2\,a\,b^2 c^2 = 6\,a^{4-1} b^{2-2} c^{3-2} = 6\,a^3 b^0 c^1 = 6\,a^3 1\,c^1 = 6\,a^3 c.$$

V. Divide $3\,a^2 b$ by $4\,a\,x\,y$.

As there are factors in the divisor not found in the dividend, we express the division in the fractional form, and then cancel factors common to dividend and divisor.

$$3\,a^2 b \div 4\,a\,x\,y = \frac{3\,a^2 b}{4\,a\,x\,y} = \frac{3\,a\,b}{4\,x\,y}.$$

NOTE. No examples of negative exponents are given here, as it seems better to defer a full treatment of the theory of exponents till later.

84. From these examples may be derived a

Rule for the Division of Monomials.

Divide the co-efficient of the dividend by that of the divisor for the co-efficient of the quotient.

With this co-efficient write all the literal factors of the dividend, each with an exponent equal to the excess of its exponent in the dividend above its exponent in the divisor.

Make the sign of the quotient plus when the signs of the dividend and divisor are alike, and minus when they are unlike.

85. Exercises.

140. Divide $a\,m\,n$ by n.
141. Divide $-a\,m\,n$ by m.
142. Divide $a\,m\,n$ by $-a$.

143. Divide $-amn$ by $-m$.
144. Divide $12abmn$ by $6am$.
145. Divide m by $-m$; $-m$ by $-m$.
146. Divide a by 1; a by -1; $-a$ by 1; $-a$ by -1.
147. Divide $-abc$ by -1.
148. Divide $-x^5$ by -1.

What is the result of dividing any quantity by -1?

149. Divide x^5 by x^3.
150. Divide $a^5y^3z^2$ by a^3y.
151. Divide $-m^3$ by $-m$.
152. Divide $(mxy)^3$ by x^2y^3.
153. Divide $6x^3$ by $3x$.
154. Divide $-15a^2x^3y$ by $3axy$.
155. Divide $-x^5$ by $-x$.
156. Divide $2ab$ by $3a$.
157. Divide $3y^2$ by y^4.
158. Divide a^3 by $3a$.
159. Divide $5z$ by z^5.
160. Divide $3k^5$ by $5k^3$.
161. Divide $17k^2l^3m^3$ by $-3klm^2$.
162. Divide $-8a^3x^4y^3$ by $-18y^3$.
163. Divide $216a^3x^4y^3$ by $24a^3xy^2$.
164. Divide $356amx^5$ by $89ax^4$.
165. Divide $-287l^3m^3n^4$ by $41l^3m^2n$.
166. Divide x^m by x^n.
167. Divide y^m by y.
168. Divide m^{p+q} by m^p.
169. Divide a^{2n} by a^n.
170. Divide p^{2n-1} by p^2.
171. Divide x^{p+1} by x^3.
172. Divide z by z^n.

Addition and Subtraction.

86. ILLUSTRATIVE EXAMPLES. I. Find the sum of $3a^2x$ and $5a^2x$.

Since a^2x stands for some number, $3a^2x$ stands for three times, and $5a^2x$ for five times, that number. But the sum of three times and five times a number is eight times that number.

Hence $3a^2x + 5a^2x = 8a^2x$.

II. From $7axy$ subtract $3axy$.

Reasoning as above, we find that $7axy - 3axy = 4axy$.

III. To $7axy$ add $-3axy$.

As has been shown in Art. 65, this example is the same as the last, differently stated.

Let it be noticed that $7axy - 3axy$ expresses the *difference* between $+7axy$ and $+3axy$, but the *sum* of $+7axy$ and $-3axy$.

IV. What is the sum of $-4ab^2$, $3ab^2$, $-5ab^2$, and ab^2?

As in the foregoing examples, we have only to find the algebraic sum of the co-efficients, which is -5, and with it write the literal factors that are common to the monomials added.

The work may be conveniently written as in the margin.

$$\begin{array}{r} -4ab^2 \\ +3ab^2 \\ -5ab^2 \\ +ab^2 \\ \hline -5ab^2. \end{array}$$

V. Find the sum of $5a$ and $3b$; also the sum of $6y$ and $-5z$.

Since a and b may be different numbers, the reasoning used in the foregoing examples does not apply. The sum can only be represented by writing $5a + 3b$. The sum of $6y$ and $-5z$, for the same reason, is represented by writing $6y - 5z$.

VI. Find the sum of a^3, $-3a^2b$, $3ab^2$, and $-b^3$.

This sum is represented by

$$a^3 - 3a^2b + 3ab^2 - b^3.$$

VII. From $3a$ subtract $-4b$.

This remainder is represented by changing the sign of the subtrahend, and then writing it after the minuend; thus, $3a + 4b$.

87. From the foregoing examples it is plain that addition and subtraction can be performed only when the given monomials agree in having the same literal factors. Monomials which have the same literal factors are ***similar monomials.***

Thus $3ab^2$ and $-4ab^2$ are similar; but $3ab^2$ and $3mb^2$ are not similar; neither are $3ab^2$ and $3a^2b$; neither are $4a^3$ and $4a^2$.

88. From the illustrations above given may be derived the following

Rules for the Addition of Monomials.

If the given monomials are similar: *Find the algebraic sum of the co-efficients, and with it write the literal factors common to the monomials.*

If the given monomials are dissimilar: *Write them one after the other, each with its positive or negative sign.*

NOTE. The process of uniting terms, as explained in Arts. 11–13, comes under the first of these rules.

Rule for the Subtraction of Monomials.

Reverse the sign of the subtrahend, and then apply one of the rules for addition.

89. Exercises.

173. To axy add $5axy$.
174. To $-lm$ add $4lm$.
175. To $3k^2l^3m$ add $-2k^2l^3m$.
176. To a^3 add a^3.
177. To y^2 add y^2 and y^2 and y^2.
178. From $5ax^2$ subtract $3ax^2$.
179. To $5ax^2$ add $-3ax^2$.
180. Add $+3a$, $-5a$, $-7a$, and $+19a$.
181. Add $-7m^2y$, $+3m^2y$, $-5m^2y$, and $18m^2y$.
182. From the sum of $5nz$, $-4nz$, and $-3nz$, subtract the sum of $4nz$ and $-12nz$.

Add the following:

(183.)	(184.)	(185.)	(186.)
$-\ am$	$-\ 5a^2x^5$	$+3m^2y$	$-\ klm$
$-5am$	$+\ 6a^2x^5$	$-\ m^2y$	$-3klm$
$+6am$	$-17a^2x^5$	$+2m^2y$	$-6klm$
$+\ am$	$-\ 3a^2x^5$	$-3m^2y$	$-5klm$
$+4am$	$+\ a^2x^5$	$+7m^2y$	$+5klm$

187. To x add y.
188. To x add $-y$.
189. From x subtract $-y$.
190. To $-lm$ add $-mn$.
191. From $-lm$ subtract $-klm$.
192. To a^3 add a^2.
193. To $3y$ add y^3.
194. To $3x^3$ add $4x$ and $2x^2$.
195. To $7ax^3$ add $5a^3x$ and $3a^2x^2$.
196. From $4a$ subtract 2.
197. From a^4 subtract $4a$.
198. From m^3 subtract 3.
199. From ab subtract b.
200. From $-xy$ subtract -1.
201. From $3axy$ subtract $3x$.
202. From $4a^2x$ subtract $-2ax^2$.

Polynomials.

90. The algebraic sum of two or more monomials is a ***polynomial,*** or a quantity of several terms.

A polynomial of two terms is a ***binomial,*** and one of three terms is a ***trinomial.***

Thus $a+b$, $a-b$, and $4x-3$, are binomials; $a+b+c$ and $x^2-2xy+y^2$ are trinomials; $x^4+6a^2x^2-4ax^3+a^4-4a^3x$ is a polynomial of five terms.

91. Since the value of a sum is not changed by adding the several parts in a different order, we may arrange the terms of a polynomial in any order we please.

In working with polynomials, it is always best to keep up an orderly arrangement of terms: it facilitates work, and prevents errors. When the several terms contain different powers of the same literal factor, the terms may be arranged according to the ascending or descending powers of that factor: in other cases it is well to follow the order of the alphabet.

Thus the polynomial given at the end of the last article, by arranging its terms according to the descending powers of a, becomes

$$a^4-4a^3x+6a^2x^2-4ax^3+x^4.$$

Arranging the terms according to the descending powers of x would merely reverse this order, and give

$$x^4-4ax^3+6a^2x^2-4a^3x+a^4.$$

In either case a certain symmetry is apparent in the form of the expression.

92. If a polynomial contains similar terms, they may be united into one term by methods heretofore explained. The polynomial is thus reduced to the least possible number of terms; and, when these have been arranged in some definite order, the polynomial is said to have been reduced to its ***simplest form.***

Thus, the simplest form of

$$5a^2b+9ab^2+3b^3+3a^2b+3ab^2-5b^3-7ab^2-8a^2b$$

is

$$5ab^2-2b^3.$$

93. Exercises.

Reduce to its simplest form each of the following polynomials:

203. $3ab^2+3a^2b+b^3+a^3.$

204. $a^3+x^3+a^2x+ax^2.$

205. $m^4+n^4-m^3n-mn^3+m^2n^2.$

206. $k^4+2k^3l+k^2l^2-2k^3l-4k^2l^2-2kl^3+k^2l^2+2kl^3+l^4+4k^2l^2.$

207. $4a^2x^2-8a^3x-2ax^3+x^4+16a^4.$

208. $6y^3-4y^2-3y^4+3y-1+y^5.$

209. $x^2+x+1-x^3-x^2-x.$

Addition.

94. Illustrative Examples. I. Find the sum of $3a+2b-c$, $3c-b+2a$, and $2b-4c-a+2x$.

$$\begin{array}{rrrr} 3a & +2b & -c & \\ 2a & -b & +3c & \\ -a & +2b & -4c & +2x \\ \hline 4a & +3b & -2c & +2x \end{array}$$

The sum of these polynomials is evidently equal to the sum of all the terms composing them. This sum might be expressed by writing all the terms one after the other, each with its proper sign. If, however, the polynomial thus found contained similar terms, it could be reduced by uniting the similar terms. It will be more convenient, therefore, to arrange the polynomials that are to be added so that similar terms may be written in the same column, as shown above. By so doing we find the result at once in its simplest form.

II. Find the sum of $a+b$ and $m-n$.

As there are no similar terms in these quantities, their sum is expressed by writing the terms one after the other, each with its sign, thus $a+b+m-n$.

95. From the first example may be derived a

Rule for the Addition of Polynomials.

Arrange the polynomials so that similar terms may be written in the same column.

Find the sum of the terms in each column separately, and write this sum below with its proper sign. These results taken together make up a polynomial, which is the sum of the given polynomials.

Note. When this rule does not apply, the method of Ex. II. is to be followed.

96. Exercises.

Add

(210.)

$$\begin{array}{rrr} x & -y & -3 \\ x & +y & -19 \\ \hline \end{array}$$

(211.)

$$\begin{array}{rrrr} 5x & -3y & +4z & -7 \\ -7x & +4y & -7z & +8 \\ \hline \end{array}$$

(212.)

$$\begin{array}{rrr} 3x & +7y & -z \\ -x & -3y & +4z \\ -4x & -y & -5z \\ \hline \end{array}$$

(213.)

$$\begin{array}{rrrr} 3x & -2y & +5z & -7 \\ 5x & +3y & -2z & +6 \\ -4x & -y & +z & -3 \\ \hline \end{array}$$

214. Add $x-2y$ and $2y+x$.

215. Add $-x$ and $2x-y$.

216. Add $m-n$, $2m-3n$, and $5n-2m$.

217. Add $x-y$, $y-z$, $z-w$, and $w-x$.

218. Add $x-y-z$, $y-z-x$, and $z-x-y$.

219. Add $x^3+ax^2+a^2x+a^3$, $2a^3-3a$, $2x+4ax^2-2x^3$ and $3a^2x-3ax^2+x^3+a^3$.

220. Add $x^5+5x^4+10x^3+10x^2$, $x^4+4x^3+6x^2+4x$, and x^3+3x^2+3x+3.

221. Add $2x^3-3x+5x^2+4$, x^3-2x^2+5x-6, $4+x-2x^2--3x^3$, and $3x^2+x^3-3x+1$.

222. Add $x^2-3xy+2y^2-3x+5y-23$, $2x^2+5xy+y^2+2y-4x+4$, $y^2-xy+x^2-x-2y+3$, and $3x+2y+4xy$.

223 Add $a^2-2ab+b^2-2bc+2ca+c^2$, $b^2-2bc+c^2-2ca+2ab+a^2$, and $c^2-2ca+a^2-2ab+2bc+b^2$.

Subtraction.

97. ILLUSTRATIVE EXAMPLES. I. From a subtract $b+c$.

It is plain that a polynomial will be subtracted if each term of it is subtracted. We therefore first subtract b, and from the result subtract c, which gives $a-b-c$ for the remainder sought.

Since $a-b-c$ is the *sum* of a and $-b-c$, the remainder sought in this case may be found by reversing the signs of the subtrahend, and then adding it to the minuend.

II. From a subtract $b-c$.

First b is subtracted, and the result is $a-b$; then from this $-c$ is subtracted, and the result is $a-b+c$. See Art. 65.

By reversing the signs of the subtrahend we get $-b+c$, and this added to a gives $a-b+c$, as before.

There is another way of explaining this example, which leads to the same result. We first subtract b from a, and the remainder is $a-b$. But this remainder is as much too small as the quantity that was subtracted is too great. The quantity subtracted is too great by c. The true remainder, therefore, will be found by increasing $a-b$ by c, which gives $a-b+c$ as the true remainder. See Art. 45, Ex. 307.

III. From $\quad 2a-b-3c$ $\quad$ To $\quad 2a-b-3c$

$$\begin{array}{lrrrlrrr} \text{From} & 2a- & b- & 3c & \text{To} & 2a- & b- & 3c \\ \text{subtract} & a- & 2b+ & 2c & \text{add} & -a+ & 2b- & 2c \\ \hline \text{Remainder,} & a+ & b- & 5c. & \text{Sum,} & a+ & b- & 5c. \end{array}$$

98. From these examples may be derived a

Rule for the Subtraction of Polynomials.

Reverse the signs of all the terms of the subtrahend, and then add it to the minuend.

99. Exercises.

Subtract

(224.)

$$\begin{array}{r} x+y-7 \\ \underline{x-y-3} \end{array}$$

(225.)

$$\begin{array}{r} 7x^2-2x+4 \\ \underline{2x^2+3x-1} \end{array}$$

(226.)

$$\begin{array}{r} 2x-3y+5 \\ \underline{5x-7y-3} \end{array}$$

(227.)

$$\begin{array}{r} 4ax-3a^2+2x^2 \\ \underline{4ax-a^2+4x^2} \end{array}$$

228. From $x-1$ subtract 1.

229. From $y-b$ subtract $-2b$.

230. Subtract $n+1$ from $2n-1$.

231. Subtract k from $k-1$.

232. How much must be taken from $l+m+n$ to leave for a remainder $2l-2m$?

233. From $a+b$ take $a-b$.

234. How much greater is $6a-12b$ than $-5a-10b$?

235. What must be added to $x-y$ to make the sum 1?

236. What quantity added to $7a^2-6xy+5ab$ will make the sum $12a^2+4ab-5xy$?

237. The sum of two quantities is $a^3+3a^2x+3ax^2+x^3$. If one of them is $x^3-3x^2a+3xa^2-a^3$, what is the other?

238. If $2x^4+2y^4$ be separated into two parts of which one is $x^4-4x^3y+6x^2y^2-4xy^3+y^4$, what is the other?

239. From the sum of $4x^4 + 6x^3 + 8x^2$ and $8x^2 + 12x + 16$, take $6x^3 + 9x^2 + 12x$.

240. From $\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$ take b.

241. From $x + \frac{1}{2}y + 1$ take $\frac{1}{2}x + y + \frac{1}{2}$.

242. From $\frac{2}{3}ax - \frac{1}{3}xy + \frac{2}{3}$ take $\frac{1}{3}ax + \frac{2}{3}xy - \frac{1}{3}$.

Use of Brackets.

100. To indicate that an operation is to be performed on a polynomial as a whole, the polynomial is written within parentheses, brackets, or braces, ( ), [ ], { }. A polynomial so written is treated as a single term or a single factor.

Let $a - b + c$ and $x + y - z$ be two polynomials. We write:

To indicate their sum,

$$a - b + c + (x + y - z).$$

To indicate their difference,

$$a - b + c - (x + y - z).$$

To indicate their product,

$$(a - b + c)\,(x + y - z).$$

NOTE. The sign of multiplication is usually omitted in such cases. The brackets must be used with each factor.

To indicate their quotient,

$$(a - b + c) \div (x + y - z) \text{ or } \frac{a - b + c}{x + y - z}.$$

NOTE. In the fractional form the horizontal line takes the place of, and has the same effect as, brackets.

101. Since the sum indicated by

$$a - b + c + (x + y - z)$$

is, when the addition has been performed,

$$a - b + c + x + y - z,$$

it is plain that

When the plus-sign stands before brackets, the brackets may be removed without changing the value of the expression.

102. Since the difference indicated by

$$a - b + c - (x + y - z)$$

is, when the subtraction has been performed,

$$a - b + c - x - y + z,$$

it is plain that

When the minus-sign stands before brackets, the brackets may be removed without changing the value of the expression, PROVIDED *the signs of all the terms written within the brackets are reversed.*

NOTE. When no sign is written before the first term within the brackets, the sign plus is understood.

103. Exercises.

Rid each of the following expressions of brackets without changing its value, and then unite any similar terms there may be:

243. $3a^2 - 2ax + x^2 - (a^2 - ax + x^2)$.

244. $3a^2 - 2a + 1 - (a^2 + 2a + 3) - (2a^2 - 6a - 6)$.

245. $a + b - (2a - 3b) - (5a + 7b) - (-13a + 2b)$.

246. $a - (b - c) - (c - a) - (a - b)$.

247. $x^2 + 2x + 1 - (x^2 - 2x + 1)$.

248. $x^2 - 2x + 1 + (x^2 + 2x + 1)$.

249. Free from brackets and add $x - (y - z + w)$, $y - (z - w + x)$, and $z - (w - x + y)$.

250. Free from brackets and add $a - (b - c)$, $b - (c - a)$, and $c - (a - b)$.

251. From $a - x - (x - 2a) + 2a - x$ take $a - 2x - (2a - x) + (-2a + x)$.

252. From the sum of $a - (b - c)$ and $b - (c - a)$ take $c - (b - a)$.

Sometimes there are brackets within brackets. In such cases, it is best to remove the inmost brackets first.

Rid the following expressions of brackets, and unite terms:

253. $a - [b - (x - y)]$.

254. $a - (x - a) - [x - (a - x)]$.

255. $l - m - [k - l - (l - m)] - (2l - m - k)$.
256. $a - [b - (2b + x)] + [b - (x - 2b)]$.
257. $a - [a - b - (b - c) - (c - a)]$.
258. $7x - \{3y - [4x - (6y - 2x)]\}$.
259. $[6a - (b + c)] - [a - (b - 2c)]$.
260. $1 - [1 - (1 - x)] + [2x - (3 - x)]$

104. The converse operation, that of putting some part of a given polynomial expression into brackets, is frequently convenient. From Arts. 101 and 102 it is plain that

When brackets are introduced into a polynomial expression after a PLUS-*sign, no change of value is caused.*

When brackets are introduced after a MINUS-*sign, the signs of all the terms enclosed by them must be reversed, to prevent a change in the value of the expression.*

Thus, no change is caused in the value of the expression

$$a + b - c + d - e + f$$

by introducing brackets as follows:

$$a + b - c + (d - e + f),$$
$$a + (b - c + d - e + f),$$
$$a + (b - c) + (d - e) + f,$$
$$a + (b - c + d) - e + f, \text{ and so on;}$$

nor by introducing brackets and reversing signs as follows:

$$a + b - (c - d + e - f),$$
$$a + b - c + d - (e - f),$$
$$a + b - (c - d) - (e - f),$$
$$a - (-b + c - d + e - f), \text{ and so on.}$$

105. Exercises.

Without changing the value of the expression, write within brackets preceded by a minus-sign:

261. The last three terms of $a + b - c + d$.
262. The last two terms of $l - m + n$.
263. The last three terms of $w - x - y + z$.
264. The second and third terms of $2k + l - m + 2n$.

265. The last three terms of $a^2 - b^2 + 2bc - c^2$.

266. The second, third, and fourth terms of

$$x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1.$$

Multiplication.

106. Illustrative Examples. I. Multiply $a + b$ by 3; also $a - b$ by 3.

By actual addition it may be shown that three times $a + b$ is $3a + 3b$, and that three times $a - b$ is $3a - 3b$. But these two results may also be found by multiplying each term of the multiplicand by 3.

$$\begin{array}{r} a+b \\ a+b \\ a+b \\ \hline 3a+3b \end{array} \qquad \begin{array}{r} a-b \\ a-b \\ a-b \\ \hline 3a-3b \end{array}$$

Hence $$(a + b) \times 3 = 3a + 3b.$$
$$(a - b) \times 3 = 3a - 3b.$$

See Art. 29, Exs. 75–82.

II. Multiply $a + b$ by x; also $a - b$ by x.

The reasoning used in Ex. I. above will hold good for any number, x, as well as for 3.

Hence $$(a + b)x = ax + bx.$$
$$(a - b)x = ax - bx.$$

III. Multiply x by $a + b$.

To multiply a quantity by $a + b$, we multiply it first by a, and then by b, and add the products. The sum of these partial products will be the product sought.

Hence $$x(a + b) = ax + bx.$$

IV. Multiply x by $a - b$.

To multiply a quantity by $a - b$, we may multiply it first by a, and then by b, and subtract the latter product from the former. The remainder will be the product sought.

It is better, however, to regard this process from another point of view. If we regard the multiplier $a - b$ as the *sum* of $+a$ and $-b$, and multiply the multiplicand first by $+a$, and then by $-b$, observing the rule for signs (Art. 73), we may take the *algebraic sum* of the partial products so obtained as the product sought.

The result is the same either way.

Hence $$x(a - b) = ax - bx.$$

V. Multiply $a + b$ by $x + y$.

Here we find the product of $a + b$ by x, which is $ax + bx$, and by y, which is $ay + by$, and add them to form the product sought.

Hence $$(a + b)(x + y) = ax + bx + ay + by.$$

VI. Multiply $a - b$ by $x + y$.

Here the partial products are $ax - bx$ and $ay - by$, which are to be added.

Hence $$(a - b)(x + y) = ax - bx + ay - by.$$

VII. Multiply $a + b$ by $x - y$.

Here, as in Ex. IV. above, we may find the product of $a + b$ by x, which is $ax + bx$, and the product by y, which is $ay + by$, and subtract the latter from the former.

Or, regarding $x - y$ as the sum of $+x$ and $-y$, the partial products are (Art. 73) $ax + bx$ and $-ay - by$, which are added to give the product sought.

The result is the same either way.

Hence $$(a + b)(x - y) = ax + bx - ay - by.$$

VIII. Multiply $a - b$ by $x - y$.

Here we may find the product of $a - b$ by x, which is $ax - bx$, and by y, which is $ay - by$, and subtract the latter from the former.

Or, regarding $x - y$ as the sum of $+x$ and $-y$, the partial products are (Art. 73) $ax - bx$ and $-ay + by$, which are added to give the product sought.

Hence $$(a - b)(x - y) = ax - bx - ay + by.$$

From the results of the last four examples may be deduced the following general statement:

The product of two binomials is equal to the algebraic sum of all the monomial products obtained by multiplying each term of the multiplicand by each term of the multiplier, the sign of each monomial product being determined by the signs of the terms multiplied together to form it.

This statement is also true of the product of two polynomials; for the addition of more terms to the multiplicand or the multiplier in the above examples would not change the nature of the process at all.

IX. Multiply $x^3 - 3ax^2 + 3a^2x - a^3$ by $x^2 - ax + a^2$

$$\begin{array}{llllll}
x^3 & -3ax^2 & +3a^2x & -a^3 & & \\
x^2 & -ax & +a^2 & & & \\
\hline
x^5 & -3ax^4 & +3a^2x^3 & -a^3x^2 & & \\
 & -ax^4 & +3a^2x^3 & -3a^3x^2 & +a^4x & \\
 & & a^2x^3 & -3a^3x^2 & +3a^4x & -a^5 \\
\hline
x^5 & -4ax^4 & +7a^2x^3 & -7a^3x^2 & +4a^4x & -a^6
\end{array}$$

Here we multiply each term of the multiplicand by each term of the multiplier, and arrange the partial products, for convenience in addition, so that similar terms are written in the same column. This is easiest when the multiplicand and multiplier have been arranged according to the powers of some common literal factor, as, in this case, according to the descending powers of x. It will be noticed that the terms of each partial product and the terms of the whole product follow the same law of arrangement.

107. From these examples may be derived a

Rule for the Multiplication of Polynomials.

Multiply each term of the multiplicand by each term of the multiplier, and add the results.

108. Exercises.

267. Multiply $1 - 2ax + 3bx^2$ by $3n$.
268. Multiply $2ax + by - cz$ by $2xyz$.
269. Multiply $2ab - 3ac + 5bd$ by $-2x$.
270. Multiply $3x - 2xy + 6$ by $-xy$.
271. Multiply $-3x^2 + 4xy - 8y^2$ by -1.

What is the effect of multiplying any polynomial by -1?

272. Multiply $a + 2x$ by $a - 3x$.
273. Multiply $7x - 1$ by $5x - 4$.
274. Multiply $4ab - 2ac$ by $6ab + 3ac$.
275. Find the product of $ab - bc + ca$ and $2a - b$.
276. Find the product of $2ax + 3by$ and $4y - 3x$.
277. Find the product of $l + m - n$ and $l - m + n$.
278. Given $5x + 4y + 3z$ and $2x - 3y - 5z$ to find their product.
279. Multiply $ax - b + cy - d$ by $ax + b + cy + d$.
280. Multiply $1 + \frac{1}{2}a + \frac{1}{3}b$ by $1 - \frac{1}{2}a + \frac{1}{3}b$.

281. If $a^3 - 2x^2$ and $a^3 - x^2$ are the factors, what is the product?

282. What is the product when $4x^2 - 6x + 9$ and $3 + 2x$ are the factors?

283. What is the product of which $6x^2 + 3x - 2$ and $3x - 5$ are the factors?

284. Find $x - 3$ times $x^3 + 3x^2 + 9x + 27$.

285. Find $4 + 2x + x^2$ times $x^2 - 2x + 4$.

286. Multiply $y^4 + 3y^3 + 2y^2 + y + 1$ by $y^3 - y - 1$.

287. What is the value of $a - 1$ times $a^3 + a^2 + a + 1$?

288. Multiply $3x^2 - 5x^3 + x^4 - 4x + 6$ by $2 - 3x + x^2$.

289. Find the product of the factors $a^2 - ax + x^2$ and $a^2 + ax + x^2$.

290. Multiply $a^3 - b^3 + 3ab^2 - 3a^2b$ by $3a^2b + a^3 + b^3 + 3ab^2$.

291. Multiply together $y + 1$, $y - 2$, $y + 3$, and $y - 4$.

292. Find the continued product of $y - m$, $y + m$, $y^2 - my + m^2$, and $y^2 + my + m^2$.

293. Multiply $3x^m + x^n$ by $2x^m - 3x^n$.

294. Multiply $a^m + x^m$ by $a^m - x^m$.

295. Multiply $a^m - b^m$ by $a - b$.

296. Multiply $x^n + 5x^{n-1} + 8x^{n-2}$ by $x^2 + 2x$.

109. Illustrative Example. Find the second power of $a + b$.

$$\begin{array}{lll} a & +b & \\ a & +b & \\ \hline a^2 & +ab & \\ & +ab & +b^2 \\ \hline a^2 & +2ab & +b^2 \end{array}$$

This is to multiply together the equal factors $a + b$ and $a + b$. The result $a^2 + 2ab + b^2$ is the expression required.

The second power of $a + b$ is *indicated* by writing $(a + b)^2$. The expression just found is called the *expansion* or *development* of $(a + b)^2$. This example, therefore, may be stated thus: Expand (or develop) the expression $(a + b)^2$.

It is easy to see that any power of a binomial or of a polynomial may be found by actual multiplication. In the case of high powers, however, the work would be tedious, to escape which a horter method was invented by Sir Isaac Newton, which will be scribed hereafter.

Expand the following expressions by multiplication:

297. $(a-b)^2$.
298. $(x+1)^2$.
299. $(a+b)^3$.
300. $(a-b)^3$.
301. $(y-1)^4$.
302. $(1+a)^4$.
303. $(x+3)^3$.
304. $(a+2x)^3$.
305. $(ab-bc)^3$.
306. $(a-b+c)^2$.
307. $(a+b-c)^2$.
308. $(2m-3n)^2$.
309. $(m^2-n^3)^2$.
310. $(x^2-x+1)^2$.

Reduce each of the following expressions to its simplest form by performing the operations indicated:

311. $(a+x)(a-y)+(a-x)(a+y)$.
312. $(x+h)^2-(x-h)^2$.
313. $(x+4)^3-(x-4)^3$.
314. $(a+x)^2(a-x)^2$.
315. $(l-m+n)^2-(l+m-n)^2$.
316. $(x^2-mx+1)(x^2+mx-1)$.
317. $(x+1)(x+2)(x+3)+(x-1)(x-2)(x-3)$.
318. $(x^2+4x-1)(x^2-2x+3)^2$.
319. $(a-x)^2(a+x)+(a+x)^2(a-x)$.
320. $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d)$.
321. $(l+m+n)^2-l\ (m+n-l)-m\ (l+n-m)-n\,(l+m-n)$.
322. $(x+y+z)^2+(x+y-z)^2+(x-y+z)^2+(-x+y+z)^2$.
323. $(m+n)(m^2-mn+n^2)-(m-n)(m^2+mn+n^2)$.
324. $(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)$.
325. $(x-y-z)^2+(y-z-x)^2+(z-x-y)^2$.

Division.

110. ILLUSTRATIVE EXAMPLES. I. Divide $ax+bx-cx$ by x.

We divide each term of the dividend by x. The result, $a+b-c$, is a quantity, which, when multiplied by x, produces the dividend.

Hence $$(ax+bx-cx)\div x=a+b-c.$$

II. Divide $x^5 - 4ax^4 + 7a^2x^3 - 7a^3x^2 + 4a^4x - a^5$ by $x^3 - 3ax^2 + 3a^2x - a^3$.

When the dividend is the product of two polynomials, one of which, the divisor, is given, and the other, the quotient, is required, the process of division will depend on the process by which the product of two polynomials is found.

Referring to Art. 106, Ex. IX., we see that each term of the product is either a single monomial product, found by multiplying a term of the multiplicand by a term of the multiplier, or an algebraic sum found by adding several such products. Thus, x^5 is a term of the former kind and $+7a^2x^3$ is a term of the latter kind.

Moreover, there are always at least two terms of the product which are single monomial products, and not sums of such products.

One of these is produced by multiplying that term of the multiplicand which contains the highest power of x by that term of the multiplier which contains the highest power of x. The product of no other two terms could contain a power of x so high as this: hence this, not being similar to any other monomial product, will not be united with any other, but will of itself make a term of the general product. This term, in the example cited, is x^5, produced by multiplying x^3 by x^2.

The other monomial product which will of itself make a term of the general product is produced by multiplying together the terms containing the lowest powers of x in the multiplicand and the multiplier. These terms are $-a^3$ and $+a^2$; for, since $x^0=1$, these terms may be written without change of value, $-a^3x^0$ and $+a^2x^0$. The product of these terms, $-a^{3+2}x^{0+0}$, or $-a^5x^0$, or $-a^5$, contains a lower power of x than the product of any other two terms: hence $-a^5$ will of itself make a term of the general product.

From this and other examples of multiplication may be deduced the following general statement:

When the terms of the multiplicand, the multiplier, and the product, are arranged according to the descending (or ascending) powers of some literal factor common to them all, the first term of the product equals the first term of the multiplicand multiplied by the first term of the multiplier, *and the last term of the product equals the last term of the multiplicand multiplied by the last term of the multiplier.*

This statement will guide the process of division.

$$
\begin{array}{r|l}
\text{DIVIDEND.} & \text{DIVISOR.} \\
x^5 - 4ax^4 + 7a^2x^3 - 7a^3x^2 + 4a^4x - a^5 & x^3 - 3ax^2 + 3a^2x - a^3 \\
\cline{2-2}
x^5 - 3ax^4 + 3a^2x^3 - a^3x^2 \phantom{{}+ 4a^4x - a^5} & x^2 - ax + a^2 \text{ QUOTIENT.} \\
\cline{1-1}
- ax^4 + 4a^2x^3 - 6a^3x^2 + 4a^4x - a^5 & \text{FIRST REMAINDER.} \\
- ax^4 + 3a^2x^3 - 3a^3x^2 + a^4x \phantom{{}- a^5} & \\
\cline{1-1}
a^2x^3 - 3a^3x^2 + 3a^4x - a^5 & \text{SECOND REMAINDER.} \\
a^2x^3 - 3a^3x^2 + 3a^4x - a^5 & \\
\cline{1-1}
0 \phantom{{}- 3a^3x^2 + 3a^4x - a^5} &
\end{array}
$$

The dividend and divisor having been arranged according to the descending powers of x, we divide the highest term of the dividend, x^5, by the highest term of the divisor, x^3. The result, x^2, must, by the principle above stated, be the highest term of the quotient.

We next multiply the divisor by this term of the quotient, and subtract the result from the dividend. This removes from the dividend all that part of it which is produced by multiplying the terms of the divisor by the highest term of the quotient. The remainder is that part which is produced by multiplying the terms of the divisor by the remaining terms of the quotient; and the highest term of this remainder is the product of the highest term of the divisor by the highest among the remaining terms of the quotient. Hence the next term of the quotient, $-ax$, is found by dividing $-ax^4$ by x^3.

Multiplying the divisor by $-ax$, and subtracting the result from the first remainder, we get a second remainder, the highest term of which, divided by the highest term of the divisor, will give the third term of the quotient, a^2. When the product of the divisor by a^2 has been subtracted, the terms of the dividend are exhausted, and the process ends.

It will be seen, on comparing the three subtrahends found in this process with the three partial products found in Art. 106, Ex. IX., that they are the same, as, indeed, they ought to be, since they are obtained in the same way.

It is plain that the quotient might have been found equally well by beginning with the lowest terms of the dividend and divisor, instead of the highest.

The process of division ends when the terms of the dividend are exhausted, or when a remainder is found having no term that will contain the first term of the divisor. If there is no remainder, the division is said to be exact. If there is a remainder, the quotient may be completed by adding to it a fraction having the remainder for a numerator, and the divisor for a denominator, as in the next example.

III. Divide $x^2 + y^2$ by $x + y$.

The last remainder, $2y^2$, does not contain the first term of the divisor: hence this part of the division can only be expressed. This is done by adding to the quotient already found the fraction $\frac{2y^2}{x+y}$, just as is done in the corresponding case in arithmetic.

$$\begin{array}{r|l}
x^2+y^2 & \underline{x+y} \\
\underline{x^2+xy} & x-y+\dfrac{2y^2}{x+y} \\
-xy+y^2 & \\
\underline{-xy-y^2} & \\
2y^2 &
\end{array}$$

IV. Divide $a^6 - 2a^3x^3 + x^6$ by $a^2 - 2ax + x^2$.

This example shows how terms appear in the remainders which disappear when the divisor is multiplied by the quotient to reproduce the dividend.

$$\begin{array}{l|l}
a^6-2a^3x^3+\qquad x^6 & a^2-2ax+x^2 \\
\cline{2-2}
\underline{a^6-2a^5x+a^4x^2} & a^4+2a^3x+3a^2x^2+2ax^3+x^4 \\
\quad 2a^5x-\quad a^4x^2-2a^3x^3+x^6 & \\
\quad \underline{2a^5x-4a^4x^2+2a^3x^3} & \\
\qquad\qquad 3a^4x^2-4a^3x^3+x^6 & \\
\qquad\qquad \underline{3a^4x^2-6a^3x^3+3a^2x^4} & \\
\qquad\qquad\qquad 2a^3x^3-3a^2x^4+x^6 & \\
\qquad\qquad\qquad \underline{2a^3x^3-4a^2x^4+2ax^5} & \\
\qquad\qquad\qquad\qquad a^2x^4-2ax^5+x^6 & \\
\qquad\qquad\qquad\qquad \underline{a^2x^4-2ax^5+x^6} & \\
\qquad\qquad\qquad\qquad\qquad 0 &
\end{array}$$

111. From these examples may be derived a

Rule for the Division of Polynomials.

Arrange both dividend and divisor according to the ascending or the descending powers of some literal factor common to their terms.

Divide the first term of the dividend by the first term of the divisor, and the result is the first term of the quotient. Multiply the divisor by the first term of the quotient, and subtract the result from the dividend, taking care to *arrange the terms of the remainder in the same way that the terms of the dividend and divisor have been arranged.*

Divide the first term of the remainder by the first term of the divisor, and the result is the second term of the quotient. Multiply the divisor by this term, and subtract the result from the first remainder.

With the second remainder proceed as before, repeating these operations until there is either no remainder or a remainder which does not contain the first term of the divisor at all.

If there is a remainder at last, complete the quotient by adding to it a fraction having this remainder for a numerator and the divisor for a denominator.

112. **Exercises.**

326. Divide $a^3 x^2 - 5abx^3 + 6ax^4$ by ax^2.

327. Divide $3bc + 24abc^2 - 6b^2c^2$ by $3bc$.

328. Divide $4a^2x^2 - 8abx - 2ax$ by $-2ax$.

329. Divide $a - b + c - 2x$ by -1.

What is the effect of dividing any quantity by -1?

330. Divide $ac + ad - bc - bd$ by $a - b$.

331. Divide $a^2 - ax - 6x^2$ by $a - 3x$.

332. Divide $3x^2 + 16x - 35$ by $x + 7$.

333. Divide $6 + 3a - 2b - ab$ by $2 + a$.

334. Divide $y^4 + y^3 - 4y^2 + 5y - 3$ by $y^2 + 2y - 3$.

335. Divide $a^6 - 3a^5 + 4a^4 - 9a^3 + 2a^2 + 3a - 1$ by $a^2 - 3a + 1$.

336. Divide $6ax^3 + 10a^3x + 4a^2x^2 + 15a^4 - 3x^4$ by $3a^2 - x^2 + 2ax$.

337. Divide $2ab + 6abc - 8abcd$ by $1 + 3c - 4cd$.

338. Divide $a^6 - b^6$ by $a^3 - 2a^2b + 2ab^2 - b^3$.

339. Divide $a^2x^3 + a^5 - 2abx^3 + b^2x^3 + a^3b^2 - 2a^4b$ by $ax - bx + a^2 - ab$.

340. The product of two factors is $x^2 - x - 12$, and one of them is $x + 3$, what is the other?

341. The product of two factors is $4a^2 - 15x^2 - 4ax$, and one of them is $2a + 3x$, what is the other?

342. Separate $a^2 + 3a + 2$ into two factors, one of which is $a + 2$.

343. Separate $l^2 - m^2 + 2mn - n^2$ into two factors, one of which is $l + m - n$.

344. Separate $10x^2 - 7xy - 19xz - 12y^2 - 29yz - 15z^2$ into two factors, of which one is $2x - 3y - 5z$.

345. Separate $a^3 + b^3 + c^3 - 3abc$ into two factors, of which one is $a + b + c$.

346. Find the multiplicand when the product is $3x^4 + 14x^3 + 9x + 2$, and the multiplier $x^2 + 5x + 1$.

347. Find the divisor by which $8x^4 - 22x^3y + 43x^2y^2 - 38xy^3 + 24y^4$ must be divided to give for a quotient $4x^2 - 5xy + 6y^2$.

348. One factor of $a^2x^2 - b^2x^2 + 2bxy + 2bxz - y^2 - 2yz - z^2$ is $ax - bx + y + z$, what is the other?

349. From $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$ remove the factor $3b + 2a + c$.

350. Divide $x^4 - \frac{5}{4}x^3 + \frac{11}{8}x^2 - \frac{1}{2}x$ by $x^2 - \frac{1}{2}x$.

351. Divide $\frac{1}{3}x^4 - \frac{11}{12}x^3 + \frac{41}{8}x^2 - \frac{23}{4}x + 6$ by $\frac{2}{3}x^2 - \frac{5}{6}x + 1$.

352. Divide $\frac{3}{4}x^5 - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27$ by $\frac{1}{2}x^2 - x + 3$.

353. Divide $x^{n+3} - x^{n+2} - 7x^{n+1} + 15x^n - 7x^{n-1} - x^{n-2} + x^{n-3}$ by $x^3 - 3x^2 + 2x + 1$.

354. Divide $x^{2m} - x^{m+n} - 2x^{m+p} - 6x^{2n} + 11x^{n+p} - 3x^{2p}$ by $x^m - 3x^n + x^p$.

355. Divide $b^{2n} - b^{n+1}y^{n-1} - b^{n-1}y^{n+1} + y^{2n}$ by $b^{n+1} - y^{n+1}$.

356. Divide $a^2 - x^2$ by $a - x$.

357. Divide $a^3 - x^3$ by $a - x$.

358. Divide $a^4 - x^4$ by $a - x$.

359. Take any like powers of a and x, and divide their difference by $a - x$.

360. Divide $a^2 - x^2$ by $a + x$.

361. Divide $a^4 - x^4$ by $a + x$.

362. Divide $a^6 - x^6$ by $a + x$.

363. Take any like even powers of a and x, and divide their difference by $a+x$.

364. Divide a^3+x^3 by $a+x$.

365. Divide a^5+x^5 by $a+x$.

366. Divide a^7+x^7 by $a+x$.

367. Take any like odd powers of a and x, and divide their sum by $a+x$.

368. Divide $32x^5+243$ by $2x+3$.

369. Divide a^3+1 by $a+1$.

370. Divide $64k^3+27l^3$ by $4k+3l$.

371. Divide $1+x$ by $1-x$.

In this and the following examples the division would never terminate. The terms of each quotient, however, are governed by a law of succession, which enables us to write out any number of terms after the first few have been found.

372. Divide $1+2x$ by $1-3x$.

373. Divide $1-x$ by $1+x^2$.

374. Divide 1 by $1+x$.

375. Divide 1 by $1-x$.

376. Divide 1 by $1-2x+x^2$.

377. Divide $1-a$ by $1+a$.

378. Divide a by $1-r$.

379. Divide a by $1+r$.

380. Divide 1 by $(1+a)^2$.

381. Divide $(1+a)^2$ by $(1-a)^2$.

For more exercises in division take the products found in Art. 108, Exs. 272 to 296 inclusive, and divide each of them by either of the factors used in finding it.

SECTION IV.

FACTORS. DIVISORS. MULTIPLES.

113. In multiplication, two or more factors are given, and their product is required. Factoring is the converse of multiplication: a product is given, and the factors which compose it are required.

114. A quantity which is separable into factors other than itself and 1 is a ***composite*** quantity; one which is not separable into factors other than itself and 1 is a ***prime*** quantity.

NOTE. This use of the words *prime* and *composite* relates merely to the form of the algebraic expression, not to its value. Thus a is, algebraically speaking, a prime quantity; but the number that a represents may or may not be a prime number.

115. A quantity which is separable into two equal factors is called a ***perfect square,*** and one of the equal factors is its ***square root.***

Thus, $9a^4y^2$, being the product of $3a^2y$ and $3a^2y$, is a perfect square, and its square root is $3a^2y$; also $a^2+2ab+b^2$, being equal to $(a+b)(a+b)$, is a perfect square, and its square root is $a+b$.

Since $9a^4y^2$ is also the product of $-3a^2y$ and $-3a^2y$, another square root of $9a^4y^2$ is $-3a^2y$. Indeed, we shall see hereafter that every quantity has two square roots, one positive and one negative. This distinction, however, is not necessary to our present purpose, and we shall here consider only arithmetical square roots.

116. A quantity which is separable into three equal factors is called a ***perfect cube,*** and one of the equal factors is its ***cube root.***

Thus, $8a^6y^3$, being equal to $2a^2y \times 2a^2y \times 2a^2y$, is a perfect cube, and its cube root is $2a^2y$; also $a^3+3a^2b+3ab^2+b^3$, being equal to $(a+b)(a+b)(a+b)$, is a perfect cube, and its cube root is $a+b$.

Monomials.

117. The factors of a monomial are easily found; for the very form of the expression shows what the literal factors are, while the numerical factors are found by factoring the co-efficient, an arithmetical process.

Thus, $15a^2x^3 = 3 \times 5aaxxx$, the product of seven prime factors. If it is required to separate $15a^2x^3$ into two or more factors, these seven prime factors may be combined in two or more groups, in many ways.

For example, $15a^2x^3 = 3ax \times 5ax^2 = 3a^2 \times 5x^3 = 15a \times ax^3$
$$= 3a \times 5ax \times x^2 = a^2x \times 5x \times 3x,$$
and so on.

118. A monomial is a perfect square when its co-efficient is a square number and the exponents of the literal factors are all divisible by two.

When is a monomial a perfect cube? A perfect fourth power? Write two perfect cubes; two perfect squares; a perfect fourth power.

119. Exercises.

1. Separate $5a^2b$ into two factors in three different ways.
2. Separate m^3x^2 into two factors in five different ways.
3. Separate $16a^6y^4$ into two equal factors.
4. What is the square root of $25a^8$?
5. What is one of the equal factors into which x^{2n} may be separated?
6. Separate $216a^6y^3$ into three equal factors.
7. What is the cube root of $1728x^{12}y^9$?
8. Separate $64a^6b^{12}$ into two equal factors, and each of these into three equal factors. One of the latter is what root of the given quantity?
9. Separate $81m^8n^4$ into four equal factors.
10. What is the fourth root of $625x^4y^8$?

11. What is the fifth root of $32\,m^5$?

12. Is a^6 a perfect square? A perfect cube? Why?

13. Is 1 a perfect square? A perfect cube? A perfect *n*th power? Why?

14. Separate $45\,m^3 y$ into three factors, two of which are equal.

15. Separate $80\,a x^3$ into two factors, one of which is a perfect square.

16. What is the greatest perfect square contained as a factor in $96\,x^3 y^7$?

17. What is the greatest perfect cube contained as a factor in $500\,a^5 b^7$?

18. Is $-a^2$ a perfect square? Can $-a^2$ be separated into two factors, one of which is a perfect square?

19. Is $-a^3$ a perfect cube? Is -1 a perfect cube?

20. Find the cube root of $-125\,k^3 l^6$.

21. Find the fifth root of $-243\,m^{20}$.

Polynomials.

120. A polynomial may or may not be separable into factors: that depends on its form. There are certain forms which indicate plainly what the factors are, and the more useful of these will be given in the following articles.

A Factor Common to all the Terms.

121. ILLUSTRATIVE EXAMPLES. I. Separate into two factors $ax + bx - cx$.

Here the factor x is found in each term of the polynomial. Dividing the polynomial by x, we find the other factor to be $a + b - c$.

Hence $$ax + bx - cx = (a + b - c)\,x.$$

II. Factor $5\,a^4 x^2 - 15\,a^3 x^3 + 20\,a^2 x^4$.

Here each term is divisible by 5, by a^2, and by x^2. Dividing the polynomial by the product of these, we find the other factor to be $a^2 - 3\,a x + 4\,x^2$.

Hence $$5\,a^4 x^2 - 15\,a^3 x^3 + 20\,a^2 x^4 = 5\,a^2 x^2\,(a^2 - 3\,a x + 4\,x^2).$$

122. Any quantity which will divide each term of a polynomial is a ***common factor*** of the terms. The greatest quantity which will divide each term is the ***greatest common factor*** of the terms.

Thus, in Ex. I above, x is a common factor of the terms; and, in Ex. II, 5, a^2, and x^2 are common factors, while $5a^2x^2$ is the greatest common factor of the terms.

In factoring a polynomial, the first step is *to take out the greatest monomial factor common to all the terms.*

123. Exercises.

Factor each of the following quantities:

22. $ab + b^2$.
23. $m^2n + mn^2$.
24. $3x^2 - 9x$.
25. $x^3 + x$.
26. $y^5 - y^3$.
27. $3k^5 + 5k^3$.
28. $5a^3x^5 + 5a^5x^3$.
29. $5x^3 + 30x^2 - 10x$.
30. $25z^2 - 10z + 5$.
31. $a^2bc - ab^2c + abc^2$.
32. $8ax^3 - 12a^2x^2 + 4a^3x$.
33. $x^4y^2 - x^3y^3 + x^2y^4$.
34. $3a^3b^2 - 6a^2b^3 + 3ab^4$.
35. $3x^5y^3 - 21x^4y^2 + 27x^3y^4$.

Polynomials of Four Terms.

124. Illustrative Examples. I. Separate into two binomial factors $ac + ad + bc + bd$.

$$ac + ad + bc + bd = a(c + d) + b(c + d)$$
$$= (a + b)(c + d).$$

We factor the first two terms by taking out the common factor a, and the last two terms by taking out the common factor b, which reduces the given polynomial to the form $a(c + d) + b(c + d)$. This quantity may be still further factored by taking out the common factor $c + d$, which reduces it to the form $(a + b)(c + d)$.

II. Factor $ac - ad - bc + bd$.

$$ac - ad - bc + bd = a(c - d) - b(c - d)$$
$$= (a - b)(c - d).$$

In this example the common factor of the last two terms is $-b$, which gives $+c - d$ to be written in the brackets.

125. From these examples we learn that a polynomial of four terms is separable into two binomial factors when it satisfies these three conditions:

(1.) *Two of the terms must have a common factor.*

(2.) *The other two terms must have a common factor.*

(3.) *When these common factors are taken out, the same binomial factor must remain in each part.*

126. Exercises.

Factor the following quantities:

36. $ax + bx + ay + by$.
37. $ax - bx + ay - by$.
38. $x^2 + ax + bx + ab$.
39. $xy - xz - y^2 + yz$.
40. $xy - 3y + 2x - 6$.
41. $a^3 + a^2 + a + 1$.
42. $6x^3 + 4x^2 - 9x - 6$.
43. $y^5 - 2y^4 - 3y^3 + 6y^2$.
44. $pqx^2 - pxy + qxy - y^2$.
45. $m^3 - mn + m^2n^2 - n^3$.

The Square of the Sum or the Difference of Two Quantities.

127. Let A and B represent any two quantities, $A + B$ their sum, and $A - B$ their difference. By actual multiplication we learn that

$$(A + B)^2 = (A + B)(A + B) = A^2 + 2AB + B^2,$$

which means that *the square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.*

Likewise we learn that

$$(A - B)^2 = (A - B)(A - B) = A^2 - 2AB + B^2,$$

which means that *the square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

128. Reversing these two formulas, we have

$$A^2 + 2AB + B^2 = (A + B)(A + B) = (A + B)^2,$$
$$A^2 - 2AB + B^2 = (A - B)(A - B) = (A - B)^2,$$

which mean that *a trinomial is a perfect square, or is separable into two equal factors, if two of its terms are perfect squares, and the other equal to double the product of their square roots.*

129. **Exercises.**

Separate into two factors

46. $x^2 + 2ax + a^2$.
47. $x^2 - 2ax + a^2$.
48. $x^2 + 6x + 9$.
49. $x^2 - 12x + 36$.
50. $x^2 - 26x + 169$.
51. $y^2 + 2y + 1$.
52. $z^2 - 40z + 400$.
53. $4x^2 + 60x + 225$.
54. $x^2 + 10xy + 25y^2$.
55. $25m^2 - 60my + 36y^2$.
56. $a^4 + 2a^2x^3 + x^6$.
57. $x^6 + 24x^3 + 144$.
58. $x^{10} - 64x^5 + 1024$.
59. $(a + x)^2 + 2m(a + x) + m^2$.
60. $x^6 + 2x^3y^5 + y^{10}$.
61. $x^{2m} + 2x^my^n + y^{2n}$.

Trinomials of the Form $x^2 + Ax + B$.

130. By actual multiplication we find that

$$(x+4)(x+7) = x^2 + 11x + 28 \qquad (x-4)(x+7) = x^2 + 3x - 28$$
$$(x-4)(x-7) = x^2 - 11x + 28 \qquad (x+4)(x-7) = x^2 - 3x - 28$$

These results are all of the form $x^2 + Ax + B$, where A and B represent integral numbers either positive or negative.

Hence it appears that *a trinomial of the form* $x^2 + Ax + B$ *is separable into binomial factors if* A *is the algebraic sum of two integral numbers, and* B *is the algebraic product of the same two numbers.*

131. Illustrative Examples. I. Separate into two factors $x^2 + 7x + 12$.

Here we seek two numbers whose product is 12 and whose sum is 7. This is done by separating 12 into pairs of integral factors, thus, $12 = 1 \times 12 = 2 \times 6 = 3 \times 4$, from which we select 3 and 4, a pair whose sum is 7.

Hence $\qquad x^2 + 7x + 12 = (x + 3)(x + 4)$.

II. Factor $x^2 - 9x + 20$.

$$20 = 1 \times 20 = 2 \times 10 = 4 \times 5.$$

Since the product is $+20$, the factors must have *like* signs; and, since their sum is -9, both must be negative. We find that $-4 - 5 = -9$.

Hence $\qquad x^2 - 9x + 20 = (x - 4)(x - 5)$.

III. Factor $x^2+7x-60$.

$$60=1\times 60=2\times 30=3\times 20=4\times 15=5\times 12=6\times 10.$$

Since the product is -60, the factors must have *unlike* signs; and, since their sum is $+7$, the greater of them must be positive. We find $-5+12=+7$.

Hence $$x^2+7x-60=(x-5)(x+12).$$

IV. Factor $x^2-5x-66$.

$$66=1\times 66=2\times 33=3\times 22=6\times 11.$$

Since the product is -66, the factors must have unlike signs; and, since their sum is -5, the greater of them must be negative. We find $+6-11=-5$.

Hence $$x^2-5x-66=(x+6)(x-11).$$

132. Exercises.

Separate into two binomial factors

62. $x^2+11x+30$.
63. $x^2-11x+24$.
64. $x^2-9x-36$.
65. x^2+x-56.
66. x^4+3x^2-88.
67. z^6+4z^3+3.
68. y^6+8y^3-9.
69. $48-14x+x^2$.
70. $56-15y+y^2$.
71. $x^2y^2-27xy+26$.
72. $y^2z^2+12yz+32$.
73. $a^4y^6-13a^2y^3+36$.
74. $m^2+9mn+8n^2$.
75. $x^2+(a+c)x+ac$.
76. $x^2-(a-c)x-ac$.
77. $(a+x)^2+3(a+x)+2$.

The Difference of the Squares of Two Quantities.

133. We learn by multiplication that

$$(A+B)(A-B)=A^2-B^2,$$

which means that *the product of the sum and the difference of two quantities is equal to the difference of their squares.*

Reversing this formula, we have

$$A^2-B^2=(A+B)(A-B),$$

which means that *the difference of two perfect squares is separable into two factors, one being the sum and the other the difference of their roots.*

134. **Exercises.**

Separate into two factors

78. $a^2 - x^2$.
79. $a^2 - 1$.
80. $1 - a^2$.
81. $4x^2 - 9y^2$.
82. $9y^2 - 16$.
83. $a^4 - b^4$.
84. $a^6 - b^6$.
85. $x^4 - 1$.
86. $1 - y^4$.
87. $m^8 - n^8$.
88. $4k^4 - 16l^2m^2$.
89. $1 - a^8x^8$.
90. $a^2 - (b-c)^2$.
91. $(a+x)^2 - (b+y)^2$.
92. $x^2 + 2xy + y^2 - z^2$.
93. $x^2 + y^2 - 2xy - 1$.
94. $a^2 - b^2 + 2bc - c^2$.
95. $2mn + l^2 - m^2 - n^2$.
96. $2x - 1 - x^2 + x^4$.
97. $2ab + a^2 - 1 + b^2$.

The Sum or the Difference of like Powers of Two Quantities.

135. Is $a^n - b^n$ exactly divisible by $a - b$?

$$
\begin{array}{l|l}
a^n - b^n & a - b \\
\cline{2-2}
\underline{a^n - a^{n-1}b} & a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots
\end{array}
$$

$$
\begin{array}{ll}
\quad a^{n-1}b - b^n \ldots & \text{1st REMAINDER.} \\
\quad \underline{a^{n-1}b - a^{n-2}b^2} & \\
\qquad a^{n-2}b^2 - b^n \ldots & \text{2d REMAINDER.} \\
\qquad \underline{a^{n-2}b^2 - a^{n-3}b^3} & \\
\qquad\quad a^{n-3}b^3 - b^n \ldots & \text{3d REMAINDER.} \\
\qquad\quad \underline{\ldots\ldots} & \\
\qquad\quad a^{n-n}b^n - b^n \ldots & n\text{th REMAINDER.}
\end{array}
$$

To answer this, we divide in the usual way, going as far as the third remainder. From these three remainders it is easy to see what the nth remainder will be.

In the first term of each successive remainder the exponent of a is diminished by 1, it being $n-1$ in the first remainder, $n-2$ in the second, $n-3$ in the third, and so on; while the exponent of b is increased by 1, it being 1 in the first remainder, 2 in the second, 3 in the third, and so on. In the first term of the nth remainder, therefore, a must have the exponent $n-n$, and b the exponent n. But $a^{n-n} = a^0 = 1$. Hence the first term of the nth remainder is b^n simply; and, as the second term is $-b^n$, the value of the nth remainder is 0.

Hence $a^n - b^n$ is exactly divisible by $a - b$.

136. Is $a^n - b^n$ exactly divisible by $a + b$?

$$\begin{array}{l|l}
a^n - b^n & a + b \\
\underline{a^n + a^{n-1}b} & a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \dots \\
\quad -a^{n-1}b - b^n \quad \dots \text{ 1st REMAINDER.} & \\
\quad \underline{-a^{n-1}b - a^{n-2}b^2} & \\
\qquad\qquad a^{n-2}b^2 - b^n \quad \dots \text{ 2d REMAINDER.} & \\
\qquad\qquad \underline{a^{n-2}b^2 + a^{n-3}b^3} & \\
\qquad\qquad\qquad -a^{n-3}b^3 - b^n \quad \dots \text{ 3d REMAINDER.} & \\
\qquad\qquad\qquad \underline{\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot} & \\
\qquad\qquad\qquad \pm a^{n-n}b^n - b^n \quad \dots \text{ nth REMAINDER.} &
\end{array}$$

From the first three remainders we see that the nth remainder must be $\pm a^{n-n}b^n - b^n$; or, since $a^{n-n} = a^0 = 1$, simply $\pm b^n - b^n$. The sign ± here used is read "plus or minus," and means that the sign of the first term is not determined: it may be plus, or it may be minus, we cannot yet tell which.

Now, it is easy to see, that, if the division were continued, the 1st, 3d, 5th, &c., remainders would each have the first term negative, while the 2d, 4th, 6th, &c., would each have the first term positive. If, therefore, n is an odd number, the nth remainder is $-b^n - b^n$, or $-2b^n$; but, if n is an even number, the nth remainder is $+b^n - b^n$, or 0.

Hence $a^n - b^n$ is exactly divisible by $a + b$ when n is an even number, but not otherwise.

137. Is $a^n + b^n$ exactly divisible by $a + b$?

By writing out the work as far as the third remainder, it will be seen that the nth remainder is $-b^n + b^n$, or 0, when n is an odd number, and $+b^n + b^n$, or $2b^n$, when n is an even number.

Hence $a^n + b^n$ is exactly divisible by $a + b$ when n is an odd number, but not otherwise.

138. Is $a^n + b^n$ exactly divisible by $a - b$?

The investigation of this case is left as an exercise.

139. The results of the last four articles may be summed up thus:

If n is a positive integer,

$a^n + b^n$ is exactly divisible by $a + b$ when n is odd,
by $a - b$ never;

$a^n - b^n$ is exactly divisible by $a + b$ when n is even,
by $a - b$ always.

When the divisor is $a+b$, the terms of the quotient are alternately positive and negative, and when the divisor is $a-b$, the terms of the quotient are all positive.

140. **Exercises.**

Write out, without actually dividing, the quotients indicated by the following expressions:

98. $\frac{a^7-b^7}{a-b}$.
99. $\frac{a^8-x^8}{a+x}$.
100. $\frac{x^9+y^9}{x+y}$.
101. $\frac{a^5-1}{a-1}$.
102. $\frac{1+x^7}{1+x}$.
103. $\frac{m^6-1}{m+1}$.
104. $\frac{1-k^8}{1-k^2}$.
105. $\frac{1-l^6m^6}{1+lm}$.
106. $\frac{m^6+n^6}{m^2+n^2}$.
107. $\frac{m^{10}+n^{10}}{m^2+n^2}$.
108. $\frac{x^9-y^9}{x^3-y^3}$.
109. $\frac{x^9+y^9}{x^3+y^3}$.

110. Separate a^4-x^4 into three factors.
111. Separate y^4-1 into three factors.
112. Separate $81-x^4$ into three factors.
113. Separate $64-x^6$ into four factors.
114. Separate c^8-z^8 into four factors.
115. Separate m^8-1 into four factors.
116. Separate a^6+x^6 into two factors.
117. Separate $a^{12}+y^{12}$ into two factors.
118. Separate m^9+n^9 into three factors.
119. Separate y^9-b^9 into three factors.

Greatest Common Divisor.

141. What is the greatest quantity which will exactly divide each of the quantities $14a^3b^2$, $6a^2bx^2$, and $18a^3b^2xy$?

In the first place, it is plain that these quantities may be exactly divided by any prime factor common to them, as 2, or a, or b. They may likewise be divided by a product of any two or more of these prime factors, as $2a$, ab, or a^2.

Taking the prime factors of one of the quantities, $14a^3b^2$, which is equal to $2\times 7aaabb$, we find by trial that 2 is a common factor of all the given quantities; 7 is not; a^3 is not, but a^2 is; b^2 is not, but b is. Hence $2a^2b$ contains all the prime factors common to the given quantities.

If we divide the given quantities by $2a^2b$, we find that the quotients $7ab$, $3x^2$, and $9abxy$, have no common factor.

Hence $2a^2b$ is the greatest quantity which will exactly divide each of the given quantities.

142. A ***common divisor*** of two or more quantities is any quantity which will exactly divide each of them. It is another name for common factor.

Quantities that have no common factor or common divisor are said to be ***prime to each other.***

143. The ***greatest common divisor*** of two or more quantities is the greatest quantity which will exactly divide each of them.

The abbreviation g.c.d. is used for greatest common divisor.

144. *The greatest common divisor of two or more quantities is the product of all the prime factors that are common to the quantities.*

145. When the given quantities are not easily separable into factors, the greatest common divisor, if one exists, may be found by a method based upon this general principle:

The greatest common divisor of two quantities is equal to the greatest common divisor of the smaller of them and the remainder found after dividing the greater by the smaller.

Note. This proof, being somewhat difficult for a beginner, is perhaps best omitted until the final review.

Proof. Let A and B stand as abbreviations for any two quantities. Divide A by B, and represent the quotient by m, and the remainder by R. We are to prove that the greatest common divisor of A and B is the same as the greatest common divisor of B and R. In other words, letting

$$\begin{array}{lll} B) & A & (m \\ & \underline{mB} & \\ & R & \end{array}$$

$D =$ greatest common divisor of A and B,

$E =$ greatest common divisor of B and R,

we are to prove that $D = E$.

First Step. *To prove that D is a factor of R.*

We know, from the way in which a remainder is found, that

$$R = A - mB.$$

Now D, which is the greatest common divisor of A and B, will exactly divide A, and exactly divide B or mB. Hence D will exactly divide the quantity $A - mB$. But $A - mB$ is equal to R. Hence D will exactly divide R; that is, D is a factor of R.

Second Step. *To prove that D cannot be greater than E.*

We know that D is a factor of B, and we have just proved that D is a factor of R. Hence D is a common factor of B and R. But no common factor of B and R can be greater than their greatest common divisor, E. Hence D cannot be greater than E.

Third Step. *To prove that E is a factor of A.*

Since a dividend is equal to the product of the divisor by the quotient plus the remainder, we have

$$A = mB + R.$$

Now E, which is the greatest common divisor of B and R, will exactly divide B or mB, and exactly divide R. Hence E will exactly divide the quantity $mB + R$. But $mB + R$ is equal to A. Hence E will exactly divide A; that is, E is a factor of A.

Fourth Step. *To prove that E cannot be greater than D.*

We know that E is a factor of B, and we have just proved that E is a factor of A. Hence E is a common factor of A and B. But no common factor of A and B can be greater than their greatest common divisor, D. Hence E cannot be greater than D.

Conclusion. We have now proved that

D cannot be greater than E,
and E cannot be greater than D.

Therefore $D = E$, *which was to be proved.*

146. Illustrative Examples. I. Find the greatest common divisor of 52 and 91.

We divide 91 by 52, and find the remainder 39. Hence the g.c.d. of 91 and 52 is equal to the g.c.d. of 52 and 39. Dividing 52 by 39, we find the remainder 13. Hence the g.c.d. of 52 and 39 is equal to the g.c.d. of 39 and 13. Dividing 39 by 13, we find no remainder.

$$
\begin{array}{llll}
52\,)\,91\,(\,1 & & & \\
52 & & & \\
\overline{39}\,)\,52\,(\,1 & & & \\
39 & & & \\
\overline{13}\,)\,39\,(\,3 & & & \\
39 & & & \\
\overline{0} & & &
\end{array}
$$

Hence $13 =$ g.c.d. of 13 and 39
$=$ g.c.d. of 39 and 52
$=$ g.c.d. of 52 and 91.

II. Find the greatest common divisor of $x^4 - x^3 + x - 10$ and $x^3 - 3x - 2$.

The first remainder, $3x^2 - 12$, contains a factor, 3, which is evidently not a factor of either of the given quantities, and consequently no part of their g.c.d. Hence this factor may be rejected, and the remaining factor, $x^2 - 4$, taken as the divisor in the second step.

FIRST STEP.

$$\begin{array}{r|l} x^4 - x^3 + x - 10 & x^3 - 3x - 2 \\ \underline{x^4 - 3x^2 - 2x} & x - 1 \\ -x^3 + 3x^2 + 3x - 10 & \\ \underline{-x^3 + 3x + 2} & \\ 3x^2 - 12 & \\ x^2 - 4 & \end{array}$$

SECOND STEP.

$$\begin{array}{r|l} x^3 - 3x - 2 & x^2 - 4 \\ \underline{x^3 - 4x} & x \\ x - 2 & \end{array}$$

THIRD STEP.

$$\begin{array}{r|l} x^2 - 4 & x - 2 \\ \underline{x^2 - 2x} & x + 2 \\ 2x - 4 & \\ \underline{2x - 4} & \\ 0 & \end{array}$$

Ans. $x - 2$.

III. Find the greatest common divisor of $2x^3 + x^2 - 8x + 5$ and $7x^2 - 12x + 5$.

We may multiply either of these quantities by any quantity that is not a factor of the other. This would not introduce a common factor, and hence would not affect the g.c.d. of the given quantities. In this example we multiply the first quantity by 7, to make the first term of it divisible by the first term of the other.

FIRST STEP.

$$\begin{array}{r|l} 14x^3 + 7x^2 - 56x + 35 & 7x^2 - 12x + 5 \\ \underline{14x^3 - 24x^2 + 10x} & 2x + 4 \\ 31x^2 - 66x + 35 & \\ \underline{28x^2 - 48x + 20} & \\ 3x^2 - 18x + 15 & \\ x^2 - 6x + 5 & \end{array}$$

SECOND STEP.

$$\begin{array}{r|l} 7x^2 - 12x + 5 & x^2 - 6x + 5 \\ \underline{7x^2 - 42x + 35} & 7 \\ 30x - 30 & \\ x - 1 & \end{array}$$

THIRD STEP.

$$\begin{array}{r|l} x^2 - 6x + 5 & x - 1 \\ \underline{x^2 - x} & x - 5 \\ -5x + 5 & \\ \underline{-5x + 5} & \\ 0 & \end{array}$$

Ans. $x - 1$.

IV. Find the greatest common divisor of $36a^6 - 18a^5 - 27a^4 + 9a^3$ and $27a^5b^2 - 18a^4b^2 - 9a^3b^2$.

These two quantities have a common monomial factor, $9a^3$. Dividing by this, the quotients are $4a^3 - 2a^2 - 3a + 1$ and $3a^2b^2 - 2ab^2 - b^2$. If the given quantities have any more common factors they will be found in the g.c.d. of these two quotients. Removing from the second quotient the factor b^2, which can form no part of this g.c.d., we have finally $4a^3 - 2a^2 - 3a + 1$ and $3a^2 - 2a - 1$, of which to find the g.c.d. Doing this in the way already explained, we find the result $a - 1$.

Hence the g.c.d. of the given quantities is $9a^3(a - 1)$.

147. Exercises.

Find, by Art. 144, the greatest common divisor

120. Of apx^2 and a^2px.
121. Of $3a^4x^5y$ and $6a^2bx$.
122. Of $7a^2b$ and $35a^4b^3x$.
123. Of abd, acd, and bcd.
124. Of $14a^2 - 7ab$ and $10ac - 5bc$.
125. Of $ax + bx$ and $ay + by$.
126. Of $ax + x^2$ and $abc + bcx$.
127. Of $a^3 - ab^2$ and $ab^2 + b^3$.
128. Of $2a^3 - 2ab$ and $5a^3 - 5ab$.
129. Of $(m + n)^2$ and $m^2 - n^2$.
130. Of $6x^2 - 12x + 6$ and $3x^2 - 3$.

Find by Art. 144, and again by Art. 145, the greatest common divisor

131. Of $x^2 - 7x + 10$ and $4x^3 - 25x^2 + 20x + 25$.
132. Of $9x^3 + 53x^2 - 9x - 18$ and $x^2 + 11x + 30$.
133. Of $x^3 - 2x - 1$ and $x^2 + 2x + 1$.
134. Of $x^2 + 5x + 4$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$.
135. Of $x^2 + (a + c)x + ac$ and $x^2 + (b + c)x + bc$.
136. Of $a^4 - x^4$ and $a^3 - a^2x - ax^2 + x^3$.
137. Of $a^2 + ab - 12b^2$ and $a^2 - 5ab + 6b^2$.
138. Of $x^3 - x^2 - 2x$ and $2x^3 + 3x^2 + x$.
139. Of $2a^2 + ab - b^2$ and $a^3 + a^2b - a - b$.

Find, by Art. 145, the greatest common divisor

140. Of 1995 and 1825.

141. Of 2945 and 3441.

142. Of $2x^3 - 3x^2 - 9x + 5$ and $2x^2 - 7x + 3$.

143. Of $4x^2 + 3x - 10$ and $4x^3 + 7x^2 - 3x - 15$.

144. Of $6a^2 + 7ax - 3x^2$ and $6a^2 + 11ax + 3x^2$.

145. Of $x^6 - 3x^4 + 3x^2 - 2$ and $x^4 - 3x^2 + 2$.

146. Of $x^4 + a^2x^2 + a^4$ and $x^4 + ax^3 - a^3x - a^4$.

147. Of $2x^3 + x^2 - x + 3$ and $2x^3 + 5x^2 + x - 3$.

148. Of $3x^2 + 16x - 35$ and $5x^2 + 33x - 14$.

149. Of $3a^3 - 3a^2b + ab^2 - b^3$ and $4a^2 - 5ab + b^2$.

150. Of $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$.

151. Of $3x^4 + 14x^3 + 9x + 2$ and $2x^4 + 9x^3 + 14x + 3$.

152. Of $6a^3 - 6a^2y + 2ay^2 - 2y^3$ and $12a^2 - 15ay + 3y^2$.

153. Of $48x^2 + 16x - 15$ and $24x^3 - 22x^2 + 17x - 5$.

154. Of $15a^4 + 10a^3b + 4a^2b^2 + 6ab^3 - 3b^4$ and $6a^3 + 19a^2b + 8ab^2 - 5b^3$.

155. Of $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$.

Least Common Multiple.

148. A ***multiple*** of a quantity is any product containing that quantity as a factor.

Thus, a multiple of a is $2ab$; a multiple of $a + x$ is $a^2 - x^2$; a multiple of a^3 is a^6.

149. A ***common multiple*** of two or more quantities is a product containing all those quantities as factors.

Thus, a common multiple of a and b is $2ab$; a common multiple of $4ax$, $2ay$, and $2a^2x$, is $12a^2xy^2$; another is $4a^2xy$.

150. The ***least common multiple*** of two or more quantities is the least product containing all those quantities as factors.

It is plain that the least common multiple of two or more quantities must contain all the prime factors of those quantities, and no other factors.

The abbreviation l.c.m. is used for least common multiple.

151. Illustrative Examples. I. Find the least common multiple of $4ax$, $2ay$, and $2a^2x$.

We select from the factors of these quantities 4 (which contains the other numerical factors), a^2 (which contains the other a's), x, and y. The product of these, $4a^2xy$, contains all the prime factors of each given quantity, and no other factors.

Hence l.c.m. of $4ax$, $2ay$, and $2a^2x$, is $4a^2xy$.

II. Find the least common multiple of x^2-2x-3, x^2+3x+2, and x^2+4x+3.

We factor the quantities thus:

$$x^2-2x-3=(x+1)(x-3)$$
$$x^2+3x+2=(x+1)(x+2)$$
$$x^2+4x+3=(x+1)(x+3).$$

Hence
$$\text{l.c.m.}=(x+1)(x-3)(x+2)(x+3)$$
$$=x^4+3x^3-7x^2-27x-18.$$

152. Exercises.

Find the least common multiple

156. Of ab, bc, and ca.
157. Of $8a^2$, $12a^3$, and $20a^4$.
158. Of klx, lmx, and mkx.
159. Of $ax+x^2$ and $abc+bcx$.
160. Of a^3-x^3 and a^2-x^2.
161. Of a^3-ab^2 and ab^2+b^3.
162. Of $ax+bx$ and $ay+by$.
163. Of $1-a$, $1+a$, and $1-a^2$.
164. Of $2x-1$, $2x+1$, $4x^2-1$, and $4x^2+1$.
165. Of $x-1$, x^2-1, $x-2$, and x^2-2.
166. Of $x^2+3x-70$ and $x^2-16x+63$.
167. Of m^2-4m+3 and m^2-2m-3.
168. Of x^2+5x+4, x^2+2x-8, and $x^2+7x+12$.
169. Of $4(1-x)^2$, $8(1-x)$, $8(1+x)$, and $4(1+x^2)$.
170. Of a^2-b^2, $(a-b)^2$, $(a+b)^2$, a^3-b^3, and a^3+b^3.
171. Of $(a-c)(b-c)$, $(b-a)(c-a)$, and $(c-b)(a-b)$.
172. Of $(x-2)(x-3)$, $(x-3)(4-x)$, and $(x-4)(2-x)$.

SECTION V.

FRACTIONS. RATIOS. PROPORTIONS.

153. A ***unit*** is a fixed quantity of any kind used as a standard of measurement or comparison in finding the numerical values of other quantities of the same kind.

A unit is represented by the symbol 1.

154. When a unit is divided into equal parts, and one of the parts is used as a standard of measurement or comparison, such a part is called a ***fractional unit.*** A collection of fractional units is a ***fractional number.***

Thus, in measuring the width of cloth, the unit might be, not a whole yard, but *one-eighth* of a yard: this would be a fractional unit. The width of the cloth might be *seven*-eighths of a yard: this would be a fractional number.

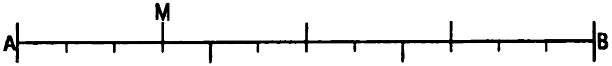
The word ***fraction*** is used to include both fractional units and fractional numbers.

155. When a unit is divided into any number, b, of equal parts, we represent *one* of the parts, the fractional unit, by $\frac{1}{b}$; *two* of the parts by $\frac{2}{b}$; *any number*, a, of the parts by $\frac{a}{b}$. Thus we see that an algebraic expression of the form $\frac{a}{b}$ may represent a fraction. The *number* represented is a, which for that reason is called the ***numerator.*** The fractional unit is $\frac{1}{b}$; and b, which gives it a *name* by showing how many such units would equal the integral unit, is called the ***denominator.***

The numerator and denominator are the *terms* of a fraction.

156. Heretofore the expression $\frac{a}{b}$ has been used to represent the quotient obtained by dividing a by b. We now see that this same expression may represent a fraction. Are these two meanings equivalent?

For example, is the quotient obtained by dividing 3 by 4 equal to the fraction $\frac{3}{4}$?



Let a straight line A B, three inches long, represent the dividend. Divide this line into four equal parts. One of these parts, A M, is the quotient. Now divide each inch into four equal parts. The quotient A M is equal to three of these parts. Hence the quotient obtained by dividing 3 inches by 4 is the fraction $\frac{3}{4}$ of an inch.

This reasoning would be the same with any other unit instead of an inch, and with any numbers instead of 3 and 4. Hence, in general,

$$a \div b = \frac{a}{b},$$

that is,

The quotient obtained by dividing a *by* b *is equal to* a *such parts as would be made by dividing* 1 *into* b *equal parts.*

157. When a quantity is divided by another quantity *of the same kind*, the quotient shows how many times, or parts of a time, the divisor is contained in the dividend. If the divisor is regarded as a unit or standard of comparison, the quotient will denote the relative magnitude of the dividend as compared with the divisor.

A number which denotes the relative magnitude of one quantity as compared with another is their ***ratio***.

158. The ratio of two quantities a and b is represented by writing $a : b$. This is read, "the ratio of a to b."

The two quantities compared are the ***terms*** of the ratio. The first term is the ***antecedent,*** and the second term is the ***consequent.***

159. The numerical value of a ratio is found by dividing the antecedent by the consequent.

Thus, 10 inches : 5 inches = 2, 5 inches : 10 inches = $\frac{1}{2}$,
3 miles : 4 miles = $\frac{3}{4}$, 7 dollars : 10 dollars = 0.7.

Hence the sign : is equivalent to a sign of division, and the expresion $a : b$ is equivalent to $a \div b$, or to $\frac{a}{b}$.

160. Any quantity expressed in the form $\frac{a}{b}$, whether it be a quotient, a fraction, or a ratio, is called an ***algebraic fraction.***

Reduction.

161. *The value of a fraction is not changed by multiplying both terms by the same quantity, nor by dividing both terms by the same quantity.*

That is, $\frac{a}{b} = \frac{am}{bm}$. For multiplying the denominator changes the fractional unit from $\frac{1}{b}$ to $\frac{1}{bm}$, thus making it $\frac{1}{m}$th as great as before; but the numerator am shows that m times as many of the smaller units are taken as of the larger; hence the value of the fraction remains the same.

Also, $\frac{ak}{bk} = \frac{a}{b}$. For dividing the denominator changes the fractional unit from $\frac{1}{bk}$ to $\frac{1}{b}$, thus making it k times as great as before; but the numerator a shows that only $\frac{1}{k}$th as many of the larger units are taken as of the smaller; hence the value of the fraction remains the same.

162. A fraction is reduced to lower terms *by dividing each term by any factor common to them*, and to its lowest terms by repeating this process until the terms have no common factor.

Since the greatest common divisor of two quantities contains all their common factors, a fraction is reduced to its lowest terms *by dividing both terms by their greatest common divisor.*

163. Exercises.

Reduce to lowest terms

1. $\frac{168}{210}$.

2. $\frac{450}{675}$.

3. $\frac{9\times8\times7\times6}{1\times2\times3\times4}$.

4. $\frac{16\times15\times14}{1\times2\times3}$.

5. $\dfrac{3a^2bc}{12ab^2c^3}$.

6. $\dfrac{4klx^2y.}{2k^2xy^2}$

7. $\dfrac{4a^3x^2y^3}{12a^2x^3y^2}$.

8. $2m:10m^2$.

9. $abc:ac$.

10. $ak:hk$.

11. $3x^3y^2:4x^2y^3$.

12. $1-x^2:1-x$.

13. $\dfrac{14m^3+21m^2}{7m^2b}$.

14. $\dfrac{mx-nx}{mnx}$.

15. $\dfrac{m^2-n^2}{3m-3n}$.

16. $\dfrac{a^2x+a^3}{ax^2-a^3}$.

17. $\dfrac{12a^2-6ab}{10ac-5bc}$.

18. $\dfrac{12m^3x^4+2m^2x^5}{18mn^2x+3n^2x^2}$.

19. $\dfrac{a^4-4x^2}{a^2-2x}$.

20. $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$.

21. $\dfrac{y^2+(k+l)y+kl}{y^2+(l+m)y+lm}$.

22. $\dfrac{x^2+2x-3}{x^2+5x+6}$.

23. $\dfrac{2m^2-3m+1}{m^2+m-2}$.

24. $\dfrac{1+y^3}{1+2y+2y^2+y^3}$.

164. Illustrative Examples. I. Change $\frac{k}{l}$ to an equivalent fraction having $a\,l\,x$ for a denominator.

To produce the denominator $a\,l\,x$, we multiply l by $a\,x$; then the numerator k must be multiplied by $a\,x$ to prevent a change in the value of the fraction.

Hence $$\frac{k}{l}=\frac{a\,k\,x}{a\,l\,x}.$$

II. Change a to an equivalent fraction having h for a denominator.

Any integral quantity may be written in the fractional form with the denominator 1. Thus, $a=\frac{a}{1}$. Now multiply both terms by h, and the result $\frac{a\,h}{h}$ is the fraction required.

165. Exercises.

25. Change $\frac{4}{9}$ to an equivalent fraction having 18 for a denominator.

26. Change $\frac{3\,m\,x}{2\,y}$ to an equivalent fraction having $2\,a\,x^2\,y$ for a denominator.

27. Change $\frac{7}{8}$ to fortieths. What is the fractional unit of the given fraction? Of the resulting fraction?

28. What expresses the value of $\frac{3}{4}$ when $\frac{1}{20}$ is the fractional unit?

29. What expresses the value of $\frac{3\,m}{8}$ when $\frac{1}{24}$ is the fractional unit?

30. What expresses the value of $\frac{a\,x}{y}$ when $\frac{1}{m\,y}$ is the fractional unit?

31. How many units of the value $\frac{1}{6}$ are there in 1? In 3?

32. How many units of the value $\frac{1}{b}$ are there in 1? In 4?

33. How many sevenths are there in $5x$?

34. How many units of the value $\frac{1}{h}$ are there in m?

35. Express the value of 4 in units of the value $\frac{1}{k}$.

36. Express the value of x in units of the value $\frac{1}{4}$.

37. Express the value of $\frac{a}{b}$ in units of the value $\frac{1}{bm}$.

38. Express $\frac{l}{x}$, $\frac{m}{y}$, and $\frac{n}{z}$ in units of the value $\frac{1}{xyz}$.

39. Express $\frac{x}{a}$, $\frac{y}{a^2}$, and $\frac{z}{ab}$ in units of the value $\frac{1}{a^2b}$.

166. The last two exercises result in fractions that have a common fractional unit, or, in other words, have a common denominator.

The quantity selected for a common denominator may be any common multiple of the given denominators; but it is usually best to select their least common multiple. The fractions will then be reduced to their ***least common denominator.***

167. Exercises.

Reduce to their least common denominator

40. $\frac{5}{7}$, $\frac{3}{5}$, and $\frac{7}{12}$.

41. $\frac{1}{7}$, 1, and $\frac{7}{3}$.

42. $\frac{a}{x}$, $\frac{b}{y}$, and $\frac{c}{z}$.

43. $\frac{2}{x+y}$ and $\frac{x-y}{xy}$.

44. $\frac{1}{1+z}$, $\frac{1}{1-z}$, and $\frac{1}{1-z^2}$.

45. $\frac{a-b}{ab}$, $\frac{a-c}{ac}$, and $\frac{b-c}{bc}$.

46. $\frac{x-3}{3}$, $\frac{x-2}{2}$, and $\frac{x-1}{1}$.

47. $\frac{a+b}{a-b}$, 1, and $\frac{a-b}{a+b}$.

48. $\frac{1}{x}$, $\frac{1}{2x}$, and $\frac{1}{3x}$.

49. $\frac{1}{x}$, $\frac{1}{x^2}$, and $\frac{1}{x^3}$.

Addition and Subtraction.

168. ILLUSTRATIVE EXAMPLES. I. Add $\frac{a}{x}$, $\frac{b}{y}$, and $\frac{c}{z}$.

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{ayz}{xyz}+\frac{bxz}{xyz}+\frac{cxy}{xyz}=\frac{ayz+bxz+cxy}{xyz}.$$

Fractions, like other quantities, can be added or subtracted only when they are expressed in units of the same value. Here the given fractions are changed to equivalent ones having the same fractional unit $\frac{1}{xyz}$; in other words, they are reduced to the common denominator xyz. When this has been done, the sum of the numerators is taken for the numerator of the result sought, and the common denominator for the denominator.

II. To c add $\frac{a}{b}$.

$$c+\frac{a}{b}=\frac{c}{1}+\frac{a}{b}=\frac{bc}{b}+\frac{a}{b}=\frac{bc+a}{b}.$$

III. From $\frac{a}{x}$ subtract $\frac{b}{y}$. To $\frac{a}{x}$ add $-\frac{b}{y}$.

$$\frac{a}{x}-\frac{b}{y}=\frac{ay}{xy}-\frac{bx}{xy}=\frac{ay-bx}{xy}.$$

Both examples have the same solution, an illustration of the fact that every example in subtraction may be changed into an example in addition by reversing the sign of the subtrahend. See Art. 65, also Art. 86.

169. The ***sign of a fraction*** is the sign + or − written before the dividing line. Besides this, the numerator and denominator have their signs, making in all three signs to be considered in connection with any fraction.

Since the value of a fraction is the quotient obtained by dividing the numerator by the denominator, we may apply the rule for signs in division (Arts. 71 and 84), and learn that

The value of a fraction is changed from positive to negative or negative to positive, (**a**) *by reversing the sign of the numerator,* (**b**) *by reversing the sign of the denomi-*

nator, (c) *by reversing the sign of the fraction; but* (d) *the value of the fraction is not changed by reversing at the same time any two of these signs.*

Thus, the value of $+\frac{+6}{+3}$ is $+2$.

But (a) $+\frac{-6}{+3}=-2$, (b) $+\frac{+6}{-3}=-2$, (c) $-\frac{+6}{+3}=-2$;

while (d) $-\frac{-6}{+3}=+2, \qquad -\frac{+6}{-3}=+2, \qquad +\frac{-6}{-3}=+2.$

What would be the effect of reversing all three signs?

170. ILLUSTRATIVE EXAMPLE. Add $\frac{a}{a-b}$ and $\frac{c}{b-a}$.

$$\frac{a}{a-b}+\frac{c}{b-a}=\frac{a}{a-b}+\frac{-c}{a-b}=\frac{a-c}{a-b}.$$

By reversing the signs of the numerator and denominator of the second fraction, it becomes $\frac{-c}{-b+a}$, or $\frac{-c}{a-b}$; but its value is unchanged. The two fractions now have a common denominator, and may be added.

171. Exercises.

Find the sum of

50. $\frac{3}{7}$, $\frac{4}{5}$, and $\frac{3}{4}$.

51. $\frac{5}{7}$, $\frac{3}{8}$, and $\frac{7}{12}$.

52. $\frac{1}{7}$, 1, and $\frac{7}{3}$.

53. $\frac{k}{p}$, $\frac{l}{q}$, and $\frac{m}{r}$.

54. $\frac{p}{ab}$, $\frac{q}{ac}$, and $\frac{r}{bc}$.

55. $\frac{1}{a}$, $\frac{1}{2a}$, and $\frac{1}{3a}$.

56. $\frac{1}{a}$, $\frac{1}{a^2}$, and $\frac{1}{a^3}$.

57. $\frac{k-l}{kl}$, $\frac{l-m}{lm}$, and $\frac{m-k}{mk}$.

58. $\frac{4x-5}{10}$, $\frac{2x}{5}$, and $\frac{7x+6}{25}$.

59. x, $\frac{3x-5}{2}$, and $\frac{2x-4}{3}$.

60. To $\frac{a+b}{2}$ add $\frac{a-b}{2}$. What results from adding half the difference of two quantities to half their sum?

61. From $\frac{a+b}{2}$ subtract $\frac{a-b}{2}$. What results from subtracting half the difference of two quantities from half their sum?

Add

62. $\frac{a}{a+b}$ and $\frac{a}{a-b}$.

63. $\frac{a}{a+b}$ and $\frac{b}{a-b}$.

64. $\frac{x+1}{x-1}$ and $\frac{x-1}{x+1}$.

65. $\frac{x-a}{x+2a}$ and $\frac{x+2a}{x-3a}$.

66. $\frac{3}{1+x^3}$ and $\frac{x-2}{1-x+x^2}$.

67. $\frac{x}{x-y}$ and $\frac{x-y}{y-x}$.

Subtract

68. $\frac{2x-3}{4}$ from $\frac{5x-1}{8}$.

69. $\frac{5x+4}{9}$ from $\frac{10x+17}{18}$.

70. $\frac{x-y}{x+y}$ from $\frac{x+y}{x-y}$.

71. $\frac{x}{x+1}$ from $\frac{3x}{x+2}$.

72. $\frac{2x}{x+y}$ from $\frac{x+y}{y}$.

73. $\frac{h-k}{k-h}$ from $\frac{k}{h-k}$.

74. Which is greater, $\frac{15}{16}$, or $\frac{21}{23}$? How much?

75. Which is greater, $\frac{1}{19}$ of 20, or $\frac{1}{20}$ of 21? How much?

Reduce to one fraction

76. $\frac{7x-10}{5}-\frac{3x-7}{6}-\frac{27x-30}{30}$.

77. $\frac{a-b}{ab}-\frac{a-c}{ac}+\frac{b-c}{bc}$.

78. $\frac{y^3}{(x+y)^3}-\frac{xy}{(x+y)^2}+\frac{x}{x+y}$.

79. $\frac{3m+2n}{3m-2n}-\frac{3m-2n}{3m+2n}$.

80. $\frac{a}{a+1}-\frac{a}{1-a}+\frac{a^2}{a^2-1}$.

81. $\frac{1}{x-1}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}$.

82. $\frac{x+y}{y}-\frac{2x}{x+y}+\frac{x^3-x^2y}{y^3-x^2y}$.

Multiplication and Division.

172. ILLUSTRATIVE EXAMPLES. I. Multiply $\frac{a}{b}$ by m.

We multiply the numerator by m. This leaves the value of the fractional unit unchanged; but there are m times as many of these units in the product as in the multiplicand.

$$\frac{a}{b} \times m = \frac{am}{b}$$

II. Multiply $\frac{a}{bm}$ by m.

We divide the denominator by m. This leaves the number of fractional units unchanged; but the unit of the product is m times as great as the unit of the multiplicand.

$$\frac{a}{bm} \times m = \frac{a}{b}$$

173. From these examples we learn that

A fraction is multiplied by multiplying its numerator, or by dividing its denominator.

How is a ratio multiplied?

174. Exercises.

83. Multiply $\frac{3}{16}$ by 5.

84. Multiply $\frac{3}{20}$ by 4.

85. Multiply 3 : 7 by 2.

86. Multiply 5 : 6 by 3.

87. Multiply $\frac{3}{10}$ by 15.

88. Multiply $\frac{6}{7}$ by 21.

89. Multiply $\frac{a}{b}$ by b.

90. Multiply $\frac{2a}{b^2}$ by $3c$.

91. Multiply $\frac{1}{z}$ by $4m$.

92. Multiply $\frac{2ab}{3xy}$ by $3y$.

93. Multiply $\frac{2uv}{5k^2l^3}$ by kl^2.

94. Multiply $\frac{ab}{cd}$ by bd.

95. Multiply $\frac{3a^2x}{7mn}$ by $2m^2x$.

96. Multiply $\frac{a+ab}{b+bc}$ by abc.

97. Multiply $\frac{a^2}{a^2-z^2}$ by $a+z$.

98. Multiply $\frac{1}{k-l}$ by $k+l$.

99. Multiply $\frac{3mn}{m+1}$ by m^2-1.

100. Multiply $\frac{y}{y^3+1}$ by y^2+y.

175. Illustrative Examples. I. Divide $\frac{am}{b}$ by m.

We divide the numerator by m. This leaves the value of the fractional unit unchanged; but there are only $\frac{1}{m}$th as many of them in the quotient as in the dividend.

$$\frac{am}{b} \div m = \frac{a}{b}$$

II. Divide $\frac{a}{b}$ by m.

We multiply the denominator by m. This leaves the number of fractional units unchanged; but the unit of the quotient is only $\frac{1}{m}$th as great as the unit of the dividend.

$$\frac{a}{b} \div m = \frac{a}{bm}$$

176. From these examples we learn that

A fraction is divided by dividing its numerator, or by multiplying its denoninator.

How is a ratio divided?

177. Exercises.

101. Divide $\frac{8}{9}$ by 4.

102. Divide $\frac{2}{9}$ by 3.
103. Divide $4:7$ by 2.
104. Divide $2:3$ by 5.

105. Divide $\frac{6}{7}$ by 9.

106. Divide $\frac{9a^2y}{4n}$ by $3ay$.

107. Divide $a + \frac{ab}{c}$ by a.

108. Divide $9:20$ by 12.
109. Divide $3x^2:4y^2$ by $3x$.
110. Divide $m^2x:k^2y$ by mk.

111. Divide $\frac{a^3+x^3}{a^2+x^2}$ by $a+x$.

112. Divide $\frac{a+b}{x-y}$ by $x+y$.

113. Divide $\frac{m^2-my}{2x}$ by $mx-xy$.

114. Divide $\frac{a^2+2ax+x^2}{a-x}$ by a^2-x^2.

115. Divide $\frac{4-4m+m^2}{2+m}$ by m^2-4.

116. Divide $\frac{x^2-3x+2}{x^2-3x+9}$ by x^2+2x-3.

To Multiply by a Fraction.

178. To multiply a quantity by a fraction $\frac{m}{n}$ is to find the $\frac{m}{n}$ part of that quantity.

179. Illustrative Examples. I. Multiply a by $\frac{m}{n}$.

The $\frac{1}{n}$ part of a is $a \div n$, or $\frac{a}{n}$; and the $\frac{m}{n}$ part is m times this, or $\frac{am}{n}$.

$$a \times \frac{m}{n} = \frac{am}{n}$$

From this example we learn that

A quantity is multiplied by a fraction by multiplying the quantity by the numerator of the fraction, and dividing the result by the denominator.

II. Multiply $\frac{a}{b}$ by $\frac{m}{n}$.

The $\frac{1}{n}$ part of $\frac{a}{b}$ is $\frac{a}{b} \div n$, or $\frac{a}{bn}$; and the $\frac{m}{n}$ part is m times this, or $\frac{am}{bn}$.

$$\frac{a}{b} \times \frac{m}{n} = \frac{am}{bn}$$

From this example we learn that

The product of two fractions is a fraction having for a numerator the product of the given numerators and for a denominator the product of the given denominators.

What is the product of two ratios?

180. Exercises.

117. Multiply 3 by $\frac{4}{5}$.

118. Multiply $\frac{3}{7}$ by $\frac{5}{8}$.

119. Multiply $\frac{4}{9}$ by $\frac{15}{16}$.

120. Multiply $2ax$ by $\frac{2x}{a}$.

121. Multiply $\frac{a}{b}$ by $\frac{b}{a}$.

122. Multiply $\frac{1}{b}$ by $\frac{1}{m}$.

123. Multiply $\frac{3a^2}{10x}$ by $\frac{5x^3}{9ay^2}$.

124. Multiply $2:7$ by $4:7$.

125. Multiply $2:3$ by $7:9$.

126. Multiply k by $a:b$.

127. Multiply $m:n$ by $p:q$.

128. Multiply $\frac{k}{k-l}$ by $\frac{l}{k+l}$.

129. Multiply $\frac{2ax}{a^2-x^2}$ by $\frac{a+x}{x}$.

130. Multiply k^4-1 by $\frac{m}{k^2-1}$.

131. Multiply $\frac{a^2+ax+x^2}{a^2-ax+x^2}$ by $\frac{a-x}{a+x}$.

132. Multiply b^6+y^6 by $\frac{1}{b^2+y^2}$.

133. Multiply x^2+x-2 by $\frac{x+1}{x+2}$.

134. Multiply $y^2-7y+10$ by $\frac{2y+5}{y-2}$.

135. Multiply $\frac{a}{b}$ by $\frac{a}{b}$; in other words, find the value of $\left(\frac{a}{b}\right)^2$.

Find the value

136. Of $\left(\frac{2x}{3y}\right)^4$.

137. Of $\left(\frac{x+y}{x-y}\right)^2$.

138. Of $\left(\frac{a}{b}+\frac{b}{a}\right)^2$.

139. Of $\left(a+\frac{1}{a}\right)^4$.

140. Of $\left(x+\frac{h}{x}\right)^2$.

141. Of $\left(x-\frac{1}{x}\right)^3$.

Multiply together

142. $\frac{m^4-n^4}{m^2-2mn+n^2}$ and $\frac{m-n}{m^2+mn}$.

143. $\frac{3by}{4cx}$, $\frac{b^2-y^2}{a^2-y^2}$, $\frac{ca+cy}{b^2+by}$, and $\frac{a-y}{b-y}$.

144. $\frac{1}{h^2}+\frac{1}{h}+1$ and h^2-h+1.

145. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $\frac{1}{a}-\frac{1}{b}+\frac{1}{c}$.

146. $\frac{l}{m}+\frac{m}{n}+\frac{n}{l}$ and $\frac{l}{n}+\frac{n}{m}+\frac{m}{l}$

To Divide by a Fraction.

181. ILLUSTRATIVE EXAMPLES. I. Divide a by $\frac{m}{n}$.

This is to ask how many times $\frac{m}{n}$ is contained in a.

$$a \div \frac{m}{n} = \frac{an}{n} \div \frac{m}{n} = an \div m = \frac{an}{m}$$

To answer this, we reduce dividend and divisor to a common denominator. Then $\frac{m}{n}$ is contained in a as many times as $\frac{m}{n}$ is contained in $\frac{an}{n}$; and this is as many times as the numerator m is contained in the numerator an, which is $\frac{an}{m}$ times.

II. Divide 1 by $\frac{m}{n}$.

Reasoning as above, $1 \div \frac{m}{n} = \frac{n}{n} \div \frac{m}{n} = n \div m = \frac{n}{m}$.

III. Divide $\frac{a}{b}$ by $\frac{m}{n}$.

Reducing the fractions to a common denominator, we reason as above.

$$\frac{a}{b} \div \frac{m}{n} = \frac{an}{bn} \div \frac{bm}{bn} = an \div bm = \frac{an}{bm}$$

182. From these examples we learn that

A quantity is divided by a fraction by multiplying the quantity by the denominator of the fraction, and dividing the result by the numerator.

183. The ***reciprocal*** of a quantity is the quotient obtained by dividing 1 by that quantity.

Thus, the reciprocal of a is $\frac{1}{a}$, the reciprocal of $\frac{1}{a}$ is a, the reciprocal of $\frac{m}{n}$ is $\frac{n}{m}$, and the reciprocal of $a:b$ is $b:a$.

If the product of two quantities is 1, then each is the reciprocal of the other.

Thus, if $xy=1$, then $x=\frac{1}{y}$, and $y=\frac{1}{x}$.

184. Since $a \div \frac{m}{n} = \frac{a\,n}{m}$, and $a \times \frac{n}{m} = \frac{a\,n}{m}$,

it follows (Axiom 7) that $a \div \frac{m}{n} = a \times \frac{n}{m}$;

that is,

Dividing by a fraction is equivalent to multiplying by the reciprocal of the fraction.

185. Exercises.

147. Divide 3 by $\frac{1}{2}$.
148. Divide 4 by $\frac{2}{3}$.
149. Divide a by $\frac{3}{8}$.
150. Divide $\frac{2}{3}$ by $\frac{4}{5}$.
151. Divide $\frac{3}{8}$ by $\frac{3}{4}$.
152. Divide $\frac{1}{7}$ by $\frac{1}{3}$.
153. Divide 5 by 2 : 3.
154. Divide 5 : 6 by 4 : 3.
155. Divide 1 by $\frac{4}{5}$.
156. Divide 1 by 7 : 10.
157. Divide $\frac{a}{b}$ by $\frac{b}{a}$.
158. Divide $\frac{1}{3}x$ by $\frac{2}{9}x$.
159. Divide $\frac{b-3a}{2ab}$ by $\frac{2a-b}{4a}$.
160. Divide $\frac{am-mx}{a+r}$ by $\frac{ak-kx}{a+r}$.
161. Divide $\frac{2-x}{y}$ by $\frac{x}{1-x}$.
162. Divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$.
163. Divide $\frac{x+a}{2x-2b}$ by $\frac{x+b}{10x+2a}$.
164. Divide $\frac{x^4-k^4}{x^2-2kx+k^2}$ by $\frac{x^2+kx}{x-k}$.
165. Find the value of $\frac{3x}{2x-2} : \frac{2x}{x-1}$.
166. Find the value of $\frac{a^3-x^3}{a^3+x^3} : \frac{a-x}{a+x}$.
167. Find the value of $\frac{x^4-y^4}{x^2-2xy+y^2} : \frac{x^2+xy}{x-y}$.

186. Miscellaneous Exercises.

Reduce by performing the operations indicated

168. $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}$.

169. $\dfrac{1}{1+a}+\dfrac{1}{1-a}$.

170. $\dfrac{1}{a-1}-\dfrac{1}{a+1}$.

171. $\dfrac{1}{1-x^2}-\dfrac{1}{1+x^2}$.

172. $\dfrac{1+x^2}{1-x^2}-\dfrac{1-x^2}{1+x^2}$.

173. $\dfrac{1+\dfrac{1}{a}}{1+\dfrac{1}{b}}$.

174. $\left(\dfrac{1}{2}+\dfrac{y-3}{2}\right)\times\left(\dfrac{1}{3}+\dfrac{y-2}{3}\right)$.

175. $\left(\dfrac{a}{b}+\dfrac{b}{2a}\right)\times\left(\dfrac{b}{a}+\dfrac{a}{2b}\right)$.

176. $\left(1-\dfrac{2a}{1+a}\right)\times\left(1+\dfrac{2a}{1-a}\right)$.

177. $\left(2+\dfrac{1}{x}\right)\div\left(1-\dfrac{2}{x}\right)$.

178. $\left(1+\dfrac{x-1}{x+1}\right)\div\left(1-\dfrac{x-1}{x+1}\right)$.

179. $\left(\dfrac{1}{1+x}\right)\div\left(1-\dfrac{1}{1+x}\right)$.

180. $1\div\left(1+\dfrac{x}{4-x}\right)$.

181. $\dfrac{x+\dfrac{2x}{x-3}}{x-\dfrac{2x}{x-3}}$.

182. $\left(a^4-\dfrac{1}{a^4}\right)\div\left(a-\dfrac{1}{a}\right)$.

183. $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}$.

184. $\dfrac{a+b}{ab}(a^2+b^2-c^2)+\dfrac{b+c}{bc}(b^2+c^2-a^2)+\dfrac{c+a}{ca}(c^2+a^2-b^2)$.

185. $\dfrac{1}{4a^3(a+x)}+\dfrac{1}{4a^3(a-x)}+\dfrac{1}{2a^2(a^2+x^2)}$.

186. $\dfrac{1}{(m-2)(m-3)}+\dfrac{2}{(m-1)(3-m)}+\dfrac{1}{(m-1)(m-2)}$.

187. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-a)(b-c)}+\dfrac{1}{(c-a)(c-b)}$.

188. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}$.

189. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-a)(b-c)}+\dfrac{c^2}{(c-a)(c-b)}$.

190. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-a)(b-c)}+\dfrac{c^3}{(c-a)(c-b)}$.

Proportions.

187. Two quantities are said to be proportional to two others when the ratio of the first two is equal to the ratio of the second two.

Thus, if two men work, one 6 days and the other 10 days, receiving, one \$9 and the other \$15, the amounts paid are proportional to the times of working, because the ratio \$9 : \$15 is equal to the ratio 6 days : 10 days.

188. An equation denoting the equality of two ratios is a ***proportion.***

Thus, $9 : 15 = 6 : 10$ is a proportion.

In general,

$$A : B = C : D, \text{ or } \frac{A}{B} = \frac{C}{D},$$

is a proportion denoting the equality of the two ratios $A : B$ and $C : D$.

The proportion is read, "A is to B as C is to D."

What is the value of the ratio $18 : 6$? Of $24 : 8$? How do these ratios compare with each other? What proportion, then, may be written?

Compare the ratio $3 : 2$ with the ratio $15 : 10$. Write the proportion.

189. The first and last terms of a proportion are called the ***extremes,*** and the second and third are called the ***means.***

What are the extremes and what the means in the proportion $A : B = C : D$?

190. If we multiply each member of

$$A : B = C : D, \text{ or } \frac{A}{B} = \frac{C}{D},$$

by BD, the product of the consequents, we get, after reducing,

$$AD = BC,$$

that is,

The product of the extremes of a proportion is equal to the product of the means.

This principle is called the ***test of proportions.***

Which of the following satisfy the test? Which do not?

a. $4:6 = 8:12$.

b. $5:7 = 15:22$.

c. $-7:21 = 11:-33$.

d. $9:12 = -18:24$.

191. If we divide each member of

$$AD = BC$$

by DB, we get, after reducing,

$$\frac{A}{B} = \frac{C}{D}, \text{ or } A:B = C:D,$$

that is,

If two products are equal, and each is composed of two factors, then the two factors of one product may be made the means, and the two factors of the other product the extremes, of a proportion.

The four factors may be arranged to form a proportion in eight different ways. For each member of the given equation may be divided not only by DB, but by DC, by AB, and by AC; and then each member of $BC = AD$ (obtained from the given equation by interchanging its members) may be divided by the same four divisors. The results reduced are,

(1) $A:B = C:D$.

(2) $A:C = B:D$.

(3) $D:B = C:A$.

(4) $D:C = B:A$.

(5) $B:A = D:C$.

(6) $C:A = D:B$.

(7) $B:D = A:C$.

(8) $C:D = A:B$.

192. Comparing (2) with (1), we learn that

The means of a proportion may be interchanged.

Comparing (3) with (1), we learn that

The extremes of a proportion may be interchanged.

Comparing (5) with (1), we learn that

The two antecedents of a proportion may be made consequents, and the two consequents antecedents.

Interchange the terms of the proportion $4:7 = 12:21$ in the ways above indicated, and test the results.

193. If any three of the terms of a proportion are known, the other may be found by applying the test.

Thus, let x represent the unknown term in the proportion

$$A : B = C : x.$$

By the test,

$$A x = B C, \text{ whence } x = \frac{B C}{A}.$$

The solution would be made in the same way if the unknown term were the third, or the second, or the first.

194. Exercises.

191. Are 16, 38, 48, and 114 proportional?

192. Are 6, 2, $\frac{1}{2}$, and $\frac{1}{6}$ proportional?

193. Given $112 : 539 = 48 : x$, to find x.

194. Given $75 : x = 48 : 176$, to find x.

195. Given $115 : 92 = x : 120$, to find x.

196. Given $x : 88 = 77 : 28$, to find x.

197. Given $\frac{1}{2} : \frac{1}{3} = \frac{1}{4} : x$, to find x.

198. On a map of the United States the distance from Washington to New York is 2.3 inches, and from New York to St. Louis, 10.6 inches. Knowing the distance from Washington to New York to be 226 miles, estimate the distance from New York to St. Louis.

199. Convert $b x = a y$ into a proportion.

200. Convert $5 x = 8 y$ into a proportion, so as to show the value of the ratio $x : y$.

201. Find the ratio of x to y from the equation $12 x = 15 y$.

202. Find the value of $x : y$ from the equation $8 x + 9 y = 6 x + 19 y$.

203. Given $x + y : x - y = 5 : 2$, to find the value of $x : y$.

204. Given $x + 3 : x + 5 = 5 : 6$, to find x.

205. Given $x - 4 : x - 8 = 3 : 2$, to find x.

206. Given $3 + x : 7 + x = 7 : 11$, to find x.

207. What number must be added to each term of the ratio $5 : 9$ to make it equal to the ratio $2 : 3$?

208. Given $x : 60 - x = 7 : 5$, to find x.

209. A boy walking from his home to the city at the rate of three miles an hour passes two guide-boards, the first after he has walked three hours, and the second after he has walked five hours. The distances to the city noted on these guide-boards are in the ratio of 11 to 9. What is the distance from the boy's home to the city?

210. Divide $1500 between two persons so that their shares shall be proportional to 2 and 3.

SOLUTION. Let x represent one share, $1500 - x$, the other; then the problem may be stated in the form,

$$x:1500-x=2:3,$$

from which the value of x may be found.

211. A rope 98 feet long is to be cut in two so that the parts may be in the ratio of 3 to 4. Where shall it be cut?

212. A furnished $8000 and B $7000 as capital in a joint venture. The total profit was $4500. Determine each partner's share of it.

213. From the opposite shores of a lake 12 miles wide two skaters start to meet near the middle. If their rates of skating are proportional to 8 and 7, how far from either shore will be their meeting-place?

195. When three quantities are such that the ratio of the first to the second is equal to the ratio of the second to the third, the second quantity is called a ***mean proportional*** between the other two.

Thus, if

$$A:B=B:C,$$

then B is a mean proportional between A and C.

196. Applying the test to this proportion, we get

$$B^2=AC,$$

whence $$B=\sqrt{AC},$$

that is,

The mean proportional between two quantities is equal to the square root of their product.

197. Exercises.

214. Is 8 a mean proportional between 4 and 16? Between 2 and 32?

215. Is 6 a mean proportional between 3 and 9? Between 4 and 8? Between 4 and 9? Between 12 and 3?

216. Given $4 : x = x : 25$, to find x.

217. Given $2 : x = x : 18$, to find x.

218. Given $\frac{1}{3} : x = x : \frac{1}{12}$, to find x.

Find the mean proportional

219. Between 3 and 27.

220. Between 24 and 6.

221. Between 9 and 25.

222. Between a^2 and b^2.

223. Between $(a+b)^2$ and $(a-b)^2$.

224. Convert $a^2 - b^2 = c^2$ into a proportion.

225. A father's age is to that of his son as his son's age is to that of his grandson. The age of the father being 81 years, and that of his grandson 16 years, what is the age of the son?

226. Find one side of a square field that contains the same quantity of land as a rectangular field 90 rods long and 40 rods wide.

198. A series of equal ratios forms a ***continued proportion.***

Thus,

$$A : B = C : D = E : F = \ldots$$

is a continued proportion.

Let r be the value of each ratio in this continued proportion. Then

$$A : B = r,\ C : D = r,\ E : F = r,\ \ldots$$

whence

$$A = rB$$
$$C = rD$$
$$E = rF$$
$$\cdots\cdots$$

Adding, we get

$$A+C+E+\ldots=r(B+D+F+\ldots)$$

whence

$$(A+C+E+\ldots):(B+D+F+\ldots)=r.$$

But $r=A:B=C:D=\ldots$

therefore

$$(A+C+E+\ldots):(B+D+F+\ldots)=A:B=C:D=\ldots$$

that is,

In a continued proportion, the sum of any number of antecedents is to the sum of the corresponding consequents as any one antecedent is to its consequent.

Apply this principle to the proportion

$$1:3=2:6=3:9=4:12=5:15=\ldots$$

and test the result.

199. By a similar course of reasoning there may be derived from the given proportion

$$A:B=C:D$$

the proportions

$$A+C:B+D=A:B,$$
$$A-C:B-D=A:B.$$

But (Axiom 7) two ratios equal to the same ratio are equal to each other.

Hence

$$A+C:B+D=A-C:B-D$$

or, by interchanging the means,

$$A+C:A-C=B+D:B-D,$$

that is,

In any proportion, the sum of the antecedents is to their difference as the sum of the consequents is to their difference.

Apply this principle to the proportion $9:6=3:2$ and test the result.

200. If this principle is applied to the proportion

$$A : C = B : D,$$

which is obtained from the given proportion by interchanging the means, the result is

$$A + B : A - B = C + D : C - D.$$

Comparing this with the given proportion, we learn that

In any proportion, the sum of the first two terms is to their difference as the sum of the last two is to their difference.

Apply this principle to the proportion $9 : 6 = 3 : 2$, and test the result.

201. From two given proportions

$$A : B = C : D \text{ and } P : Q = R : S$$

we may derive, by multiplying equal ratios by equal ratios,

$$AP : BQ = CR : DS,$$

that is,

The corresponding terms of two proportions may be multiplied together, and the results will form a proportion.

Apply this principle to the proportions $3 : 6 = 4 : 8$ and $5 : 3 = 10 : 6$, and test the result.

202. If the terms of the second proportion are the same as those of the first, the result is,

$$A^2 : B^2 = C^2 : D^2.$$

If the terms of this proportion are multiplied by the terms of the given proportion, the result is,

$$A^3 : B^3 = C^3 : D^3.$$

By repeating this process we reach the general result

$$A^n : B^n = C^n : D^n,$$

that is,

If quantities are proportional, the like powers of those quantities are proportional.

203. Exercises.

From the given proportion

$$A : B = C : D$$

derive the following proportions, and state the principle expressed by each:

227. $mA : mB = nC : nD$.

228. $mA : nB = mC : nD$.

229. $\frac{A}{m} : \frac{B}{m} = \frac{C}{n} : \frac{D}{n}$.

230. $\frac{A}{m} : \frac{B}{n} = \frac{C}{m} : \frac{D}{n}$.

231. $A + B : B = C + D : D$.

232. $A - B : B = C - D : D$.

233. $A + B : A = C + D : C$.

234. $A - B : A = C - D : C$.

235. $A^2 - B^2 : B^2 = C^2 - D^2 : D^2$.

236. If $\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = \dots$

prove that $\frac{a^2}{A^2} = \frac{a^2 + b^2 + c^2 + \dots}{A^2 + B^2 + C^2 + \dots}$

237. If $\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = \dots$

prove that $\frac{a}{A} = \frac{la + mb + nc + \dots}{lA + mB + nC + \dots}$

238. Prove that

$$A : B = \frac{1}{B} : \frac{1}{A},$$

that is, *two quantities are inversely proportional to their reciprocals.*

Variation.

204. A quantity which either increases or decreases so as to have different values at different times is said to vary, and such a quantity is called a ***variable***.

Thus, a person's age varies from day to day or from year to year; the cost of merchandise varies with the quantity; the amount of work done varies with the force employed and the time spent.

Different values of a variable quantity are usually represented by the same letter with some distinguishing marks attached. Thus, different values of A are represented by

$$A', A'', A''' \ldots$$

or by

$$A_1, A_2, A_3 \ldots$$

These expressions are read, "A prime, A second, A third," &c., or "A sub one, A sub two," &c.

NOTE. Be careful not to mistake these expressions for *powers* of A.

205. One quantity varies as another ***directly*** when the two increase or decrease together and in the same ratio.

Thus, the cost of merchandise varies directly as the quantity. If m and m' represent two quantities of the same kind of merchandise, c the cost of the first, and c' the cost of the second, then

$$c : c' = m : m'.$$

This proportion is an algebraic expression of the fact that c varies as m directly.

206. One quantity varies as another ***inversely*** when one increases and the other decreases in the same ratio.

Thus, the time required to do a given amount of work varies inversely as the number of men employed; the more men, the less time. If n and n' represent the numbers of men employed, t and t' the times required by them respectively, then

$$t : t' = n' : n,$$

an expression of the fact that t varies inversely as n.

By Article 203, Exercise 238, we know that

$$n' : n = \frac{1}{n} : \frac{1}{n'}.$$

Hence, Axiom 7,

$$t : t' = \frac{1}{n} : \frac{1}{n'},$$

from which we learn that

If a quantity varies inversely as another, it varies directly as the reciprocal of the other.

207. One quantity varies as two or more others ***jointly*** when it and the product of the others increase or decrease together and in the same ratio.

Thus, the distance gone over by a moving body varies as the time and the rate jointly. If d represents the distance gone over in the time t at the rate r, and d' the distance gone over in the time t' at the rate r', then

$$d : d' = tr : t'r'.$$

208. Exercises.

239. How many days will it take 24 men to do a piece of work, if it would take 15 men 16 days?

240. Two skaters, trying their speed, find that one can cross the pond 6 times while the other is crossing 5 times. The first can skate 9 miles an hour; how many miles an hour can the other skate?

241. Let r and r' represent the rates of two moving bodies, t and t' the times required by them to move the distance d. If the value of the ratio $r : r'$ is $7 : 9$, what is the value of the ratio $t : t'$?

242. The velocity acquired by a falling body varies directly as the time of falling. If, at the end of 2 seconds, the velocity acquired is $64\frac{1}{3}$ feet per second, what is the velocity acquired at the end of 5 seconds? At the end of 10 seconds?

243. The space fallen through by a freely falling body varies directly as the square of the time of falling. At the end of 1 second, the space fallen through is $16\frac{1}{12}$ feet. How far will the body fall in 6 seconds?

244. The bulk of a given quantity of gas varies inversely as the pressure upon it, provided the temperature is unchanged. At the top of Mount Blanc the pressure of the atmosphere on bodies is half as great as at the sea-level. If a pint of air taken at the sea-level were carried to the top of that mountain, what would be its bulk there?

245. The number of vibrations made by a vibrating string in a given time varies inversely as the length of the string. If a string 30 inches long makes 276 vibrations per second, how long is a similar string that makes 184 vibrations per second?

246. It is a principle of mechanics that the lengths of the two arms of a lever are inversely proportional to the weights they support. If a bar 8 feet long carries at one end a load of 700 pounds and at the other a load of 900 pounds, at what distance from either end of the bar must be the point of support, that the two loads may balance?

If g represents the force of gravitation between two given masses of matter, and d the distance between their centres, the law of gravitation is algebraically expressed by

$$g : g' = d'^2 : d^2 \text{ or by } g : g' = \frac{1}{d^2} : \frac{1}{d'^2},$$

which expressions mean that the force of gravitation between given masses varies inversely as the square of the distance.

247. A piece of iron hung on a spring balance appears to weigh 18 pounds. This is at the earth's surface, 4000 miles from the centre. Now suppose the spring balance and the iron hanging on it to be carried 240000 miles away from the earth's centre. What will the iron appear to weigh there?

248. The intensity of light varies inversely as the square of the distance from the source of light. Two equal lights stand before a screen, one three feet off, and the other five feet off. How much of its illumination does the screen get from each of the lights?

249. Two lights are placed on the table, and a white paper screen is so placed between them, that it receives equal illumination from each. The distances from the screen to the lights are 33 inches and 55 inches respectively. If the intensity of the nearer light is equal to that of 6 wax candles, what is the intensity of the farther light?

250. The number of beats made by a pendulum in a given time varies inversely as the square root of the length of the pendulum. In an experiment it is observed that one pendulum makes 9 beats while another makes 16. The shorter pendulum is 36 inches long. How long is the other?

SECTION VI.

GENERALIZATION.

209. Algebraic language has its most important use in the statement and solution of problems in a general manner.

This may best be illustrated by considering again some of the problems of Sect. II. By comparing these problems with one another, it will be found that there are among them some groups of similar problems, so that a certain classification might be made.

For example, a well-marked class of problems is found in Art. 35. These resemble one another in the following particulars:

(1) There is a certain amount of some kind of work to be done.

(2) There are two or more persons or working forces to do that work.

(3) The time that would be taken by each person or force separately to do the work is given.

(4) The question in every case is, How long will it take all the persons or forces, working together, to do the work?

Now the method of solving these and similar problems may be shown once for all by stating the conditions in a general way and deriving from this statement a general result. This general result may then be applied to any particular problems we wish to solve, as will be shown in the next article.

The Work of United Forces.

210. A can do a piece of work in a days, and B can do the same in b days. In how many days can both do it, working together?

SOLUTION. Let x represent the number of days required for both to do the work. Since A can do the whole work in a days, he will do the $\frac{1}{a}$ part of it in 1 day; also B will do the $\frac{1}{b}$ part of it in 1 day; and both will do the $\frac{1}{x}$ part of it. Hence

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{x}$$

$$bx+ax=ab$$

$$(a+b)x=ab$$

$$x=\frac{ab}{a+b}.$$

Ans. Both can do the work in $\frac{ab}{a+b}$ days.

APPLICATION. This general result is a formula which may be applied to any particular problem whose conditions are similar, by performing on the two numbers corresponding to a and b the operations indicated by the expression $\frac{ab}{a+b}$.

Take, for illustration, some of the problems in Art. 35. In the Illustrative Problem, p. 21, a is 5 and b is 6; and the answer will be found by dividing the product of 5 and 6 by the sum of 5 and 6. That is,

$$\frac{ab}{a+b}=\frac{5\times 6}{5+6}=\frac{30}{11}=2\tfrac{8}{11}.$$

In Ex. 149, $a=6$, $b=3$, and $\frac{ab}{a+b}=\frac{6\times 3}{6+3}=\frac{18}{9}=2.$

In Ex. 153, $a=4$, $b=5$, and $\frac{ab}{a+b}=\frac{4\times 5}{4+5}=\frac{20}{9}=2\tfrac{2}{9}.$

In Ex. 154, $a=3$, $b=-4$, and $\frac{ab}{a+b}=\frac{3\times(-4)}{3+(-4)}=\frac{-12}{-1}=12.$

Here we have merely written the particular numerical values of a and b in the places of those letters in the formula $\frac{ab}{a+b}$, and reduced the result. This process is called *substitution.*

What other problems in Sect. II. can be solved by use of this formula? Solve them by making the proper substitutions.

211. A can do a certain piece of work in a days; B,

in b days; and C, in c days. In how many days will all three do it, working together?

SOLUTION.

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{x}$$

$$bcx+cax+abx=abc$$

$$(ab+bc+ca)x=abc$$

$$x=\frac{abc}{ab+bc+ca}.$$

APPLICATION. This general result, or formula, applied to Ex. 155, p. 22, where $a=4$, $b=3$, and $c=6$, gives

$$\frac{abc}{ab+bc+ca}=\frac{4\times3\times6}{4\times3+3\times6+6\times4}=\frac{72}{12+18+24}=\frac{72}{54}=1\tfrac{1}{3}.$$

What other problems in Sect. II. can be solved by use of this formula? Solve them by making the proper substitutions.

Proportional Parts.

212. Divide a into two parts that shall be proportional to m and n.

SOLUTION.

Let $x=$ one part,

$\frac{nx}{m}=$ the other.

Then $x+\frac{nx}{m}=a$

$$mx+nx=am$$

$$(m+n)x=am$$

$$x=\frac{am}{m+n}$$

$$\frac{nx}{m}=\frac{am}{m+n}\times\frac{n}{m}=\frac{an}{m+n}.$$

ANOTHER SOLUTION.

Let $x=$ one part,

$a-x=$ the other.

Then $x:a-x=m:n$

$$nx=am-mx$$

$$mx+nx=am$$

$$(m+n)x=am$$

$$x=\frac{am}{m+n}$$

$$a-x=a-\frac{am}{m+n}=\frac{an}{m+n}.$$

The parts are $\frac{am}{m+n}$ and $\frac{an}{m+n}$.

Two different solutions are given, to show that the form of the result depends, not on the mode of solution, but on the nature of the problem.

APPLICATION. Ex. 138, p. 20, and Exs. 210, 211, 212, and 213, p. 119.

213. Divide a into parts proportional to the numbers m, m', m'', &c.

SOLUTION.

Let $x =$ the first part

$\frac{m'x}{m} =$ the second part

$\frac{m''x}{m} =$ the third part,

and so on.

The equation is,

$$x + \frac{m'x}{m} + \frac{m''x}{m} + \ldots = a,$$

by solving which the parts are found to be

$$\frac{am}{m + m' + m'' + \ldots}, \quad \frac{am'}{m + m' + m'' + \ldots}, \quad \frac{am''}{m + m' + m'' + \ldots}, \text{ \&c.}$$

APPLICATION. These results may be used as a formula for solving questions in simple partnership: indeed, by translating them into common language we get the rule for Simple Partnership as given in the arithmetics.

214. Exercises.

NOTE. These are to be solved by making the proper substitutions in the formula.

1. Three men, A, B, and C, charter a ship and fit her out for a voyage. A furnishes \$5000, B \$7500, and C \$4500. The profits of the voyage amount to \$6239. What is each man's share?

2. Four men formed a partnership for three years. A put in \$10000, B \$15000, and C \$8000; while D put in \$2000, and was allowed to reckon his knowledge and skill in the business as equivalent to \$15000 more. The gain was \$36570. Find each partner's share of it.

3. Four men ship merchandise to England by the same steamer. The value of A's merchandise is \$15000, of B's \$22500, of C's \$17250, and of D's \$18250. During a storm half of A's merchandise and the whole of B's is thrown overboard. Distribute this loss proportionally among the four shippers.

215. Find a formula for Compound Partnership.

SOLUTION. The gain or loss must be distributed in proportion both to the amounts of capital supplied by the several partners and to the times the several amounts of capital were employed in the business. Let c, c', c'', &c., denote the several amounts of capital; t, t', t'', &c., the times; and a the amount of gain or loss. The distribution of a, then, must be into parts proportional to the products $c\,t$, $c'\,t'$, $c''\,t''$, &c.

The equation being formed and solved, as in the last article, the parts are found to be

$$\frac{a\,c\,t}{c\,t + c'\,t' + c''\,t'' + \ldots}, \quad \frac{a\,c'\,t'}{c\,t + c'\,t' + c''\,t'' + \ldots}, \quad \frac{a\,c''\,t''}{c\,t + c'\,t' + c''\,t'' + \ldots}, \text{ \&c.}$$

This formula translated into common language would give a rule for Compound Partnership.

216. Exercises.

NOTE. These are to be solved by making the proper substitutions in the formula.

4. Bird, Allen, and Marston formed a partnership. Bird put in $8000 for 18 months, Allen $12500 for 20 months, and Marston $13000 for 2 years. The total gain was $35300. What was each partner's share of it?

5. Three farmers hire a mountain pasture for $500. A turns in 75 cattle for 10 weeks, B 80 cattle for 12 weeks, and C 67 cattle for 15 weeks. What should each farmer pay?

6. Four men contract to work a copper mine for one year for $35000. A furnishes 12 hands, at $1.75 a day; B 24 hands, at $1.25; C 8 hands, at $2; and D 12 horses, at $1. Distribute the $35000 proportionally among the contractors.

7. Four men formed a partnership. A put in $10000 on Jan. 1; B put in $8000 on March 1; C put in $12000 on April 1, but withdrew it Dec. 1; D put in $20000 on Dec. 1. On the last day of the year they divided a profit of $17250. What was each partner's share?

Interest, Present Worth, Discount.

217. The arithmetical rules for Interest, Present Worth, and Discount are easily expressed by formulas.

Let $p =$ the principal.
$r =$ the rate *per annum*, expressed as a decimal.
$t =$ the time in years.
$i =$ the interest.
$a =$ the amount $= p + i$.

Then the rule for Simple Interest is expressed by the formula

$$i = prt.$$

In this formula are represented four quantities, any one of which may be found when the values of the other three are given.

Thus, to find the principal, $p = \frac{i}{rt}$; to find the rate, $r = \frac{i}{pt}$; to find the time, $t = \frac{i}{pr}$.

State in common language the meaning of each one of these formulas.

The amount is found by adding the interest, prt, to the principal, p; that is,

$$a = p + prt$$

or, factoring the last member,

$$a = p\,(1 + rt).$$

In this formula are represented four quantities, any one of which may be found when the values of the other three are given.

Write out the formula for finding t, for finding r, for finding p.

When p is the quantity to be found, the question is, *What sum of money put at interest for the time* t, *at the rate* r, *will amount to* a?

The formula for Present Worth, therefore, is

$$p = \frac{a}{1 + rt}.$$

The discount, called true discount, is the difference between the amount and the present worth, that is, $d = a - p$. Substituting for p its value above given and reducing, we get, as a formula for True Discount,

$$d = \frac{art}{1 + rt}.$$

Compound Interest.

When the interest at the end of a year, half-year, or other fixed period of time, is added to the principal, the amount becoming a new principal for the next period, the interest is said to be compounded.

Let p denote the principal; r, the rate for each period of time; a_1, the amount at the end of the first period; a_2, the amount at the end of the second period; and, in general, a_n, the amount at the end of the nth period of time. Then,

$$a_1 = p(1+r)$$
$$a_2 = a_1(1+r) = p(1+r)^2$$
$$a_3 = a_2(1+r) = p(1+r)^3$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

In general, $$a_n = p(1+r)^n$$
which is the formula for Compound Interest.

218. Exercises.

NOTE. These are to be solved by making the proper substitutions in the formulas.

8. At what rate will $250 gain $25 in 1 year 4 months?

9. At what rate will $750 amount to $831.25 in 2 years 2 months?

10. How long must a note for $7200 run to amount to $9000 at 8%?

11. The use of money being worth 6 % a year, what is the present value of $50000 payable 7 years hence?

12. Find how much money paid June 1, 1880, is equivalent to $10000 paid Dec. 1, 1888, the use of money being worth 7½% a year.

13. Find the true discount of $720 due 3 years hence at 6%.

14. Find the amount of $500 at compound interest for 3 years 6 months at 8% per annum, interest being compounded semi-annually.

15. By what number must the principal be multiplied to give the amount of a sum of money at compound interest for 5 years at 6%, interest being compounded annually?

16. What problems in Sect. II. may be solved by applying the formulas for interest, present worth, and discount?

Problems of Pursuit.

219. A courier riding a miles a day left this place n days ago. He is now pursued by another, who rides b miles a day. In how many days will the second overtake the first?

Solution. Let x represent the number of days it will take the second to overtake the first. Then the first rides $n+x$ days, at the rate of a miles a day, or $a(n+x)$ miles in all. The second rides x days, at the rate of b miles a day, or bx miles in all. But the two couriers start from the same point and arrive at the same point. Hence

$$bx = a(n+x)$$
$$bx - ax = an$$
$$x = \frac{an}{b-a}.$$

Ans. In $\frac{an}{b-a}$ days.

Application. What problems in Sect. II. may be solved by substitution in this formula?

220. Two bodies move at uniform rates, r and r', after one another in the circumference of a circle of which the length is c. At first they are distant from each other by an arc of the length a. When will these bodies be together the first time, second time, and so on, supposing their motions to be undisturbed?

Ans. At the end of the time $\frac{a}{r'-r}$,

again at the end of the time $\frac{c+a}{r'-r}$,

again at the end of the time $\frac{2c+a}{r'-r}$,

and so on.

Applications. (a) At 12 o'clock the two hands of a clock are together. When and how often will these hands be together in the next twelve hours?

The circumference of a clock-face is divided into sixty equal spaces called minute-spaces. The hour-hand goes over five of these spaces in an hour, and the minute-hand goes over sixty of them. Hence, in all problems relating to the movements of clock-hands, $c = 60$, $r = 5$, and $r' = 60$. The value of a will differ in different problems. In the present case the hands are together at the start. Hence $a = 0$, and the first result, $\frac{a}{r'-r}$, becomes $\frac{0}{60-5} = \frac{0}{55} = 0$, that is, it vanishes. The next result, $\frac{c+a}{r'-r}$, becomes $\frac{60}{60-5}$, or $1\frac{1}{11}$ hours; showing that the hands will be together at the end of $1\frac{1}{11}$ hours from 12 o'clock, or at $5\frac{5}{11}$ minutes past 1 o'clock. The next result, $\frac{2c+a}{r'-r}$, becomes $\frac{120}{60-5}$, or $2\frac{2}{11}$ hours; showing that the hands will next be together at the end of $2\frac{2}{11}$ hours from 12 o'clock, or at $10\frac{10}{11}$ minutes past 2 o'clock.

Show, by substituting in the proper expressions, that the hands will next be together at $16\frac{4}{11}$ minutes past 3 o'clock, and so on, in each successive hour $5\frac{5}{11}$ minutes later.

(b) At 6 o'clock the hands of a clock are opposite to one another. At what times during the next four hours will the hands be opposite to one another?

(c) At 3 o'clock the hands of a clock are at right angles with each other. At what times during the next six hours will the hands be at right angles with each other?

221. **Miscellaneous Exercises.**

Solve the following equations:

17. $ax + bx = m$.

18. $ax - b = m$.

19. $ax + b = mx + n$.

20. $ax - \frac{ax-b}{c} = d$.

21. $\frac{x}{a} + \frac{x}{b} = 1$.

22. $\frac{ax+b}{c} = \frac{lx+m}{n}$.

23. $\frac{x}{a+b} + \frac{x}{a-b} = m$.

24. $\frac{x}{k+1} - \frac{x}{k-1} = a$.

25. $(x-a)(x+a) = a^2 - 4ax + x^2$.

26. $(k+l+x)(k+l-x) = (k-x)(l+x) - kl$.

27. $\frac{a}{x-a} - \frac{a}{x-2a} = \frac{a}{x-3a} - \frac{a}{x-4a}$.

Solve the following problems in a general manner, and then find problems in Sect. II. that may be solved by substitution in the results:

28. Separate the number a into two parts such that one part is m greater than the other.

29. Separate the number a into two parts such that one part is m times as great as the other part.

30. Separate the number a into three parts such that the second may be m times the first, and the third n times the second.

31. If a pounds of sugar, worth l cents a pound, are mixed with b pounds of another kind, worth m cents a pound, and c pounds of a third kind, worth n cents a pound, what is the value of a pound of the mixture?

32. A boy having a certain sum of money in his pocket finds that, if he buys a quantity of raisins at m cents a pound, he will have b cents left, but, if he buys the same quantity of figs at n cents a pound, he will have c cents left. How much money has he?

33. A person has a hours at his disposal. How far may he ride out in a coach that travels m miles an hour and walk back at the rate of n miles an hour?

34. A man agreed to work n days, on condition that he should receive a cents for every day he worked, but should forfeit b cents for every day he was idle. At the end of the time he received c cents. How many days did he work? How many days was he idle?

35. A person who can row r miles an hour in still water rowed down stream a certain distance and back again in a hours, the stream flowing uniformly all the time r' miles an hour. How far down did he row?

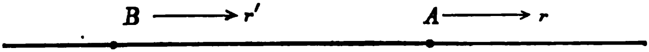
36. Show that, disregarding the days of grace, the bank discount exceeds the true discount by the $\frac{r^2t^2}{1+rt}$ part of the sum discounted.

The Symbols 0, ∞, and $\frac{0}{0}$.

222. When particular numbers are substituted for the letters in a formula, there sometimes arise peculiar results, for the purpose of illustrating which we shall solve a problem of pursuit in a general manner, and apply the result to some special cases.

PROBLEM. Two bodies, A and B, are moving in the same straight line and at the uniform rates r and r' respectively. At the present moment the distance between them is d. When will the two bodies be together?

SOLUTION. To fix our ideas, we shall assume that B is pursuing A, and that both are moving towards the right, as shown in the figure.



Let x represent the time required for B to overtake A. Then the distance gone over by B is $r'x$, and that gone over by A is rx. The equation is $r'x = d + rx$, whence $x = \frac{d}{r' - r}$.

General Answer. After the lapse of the time $\frac{d}{r' - r}$.

APPLICATIONS. (a) What does the result become when $d = 24$ miles, $r = 8$ miles an hour, and $r' = 12$ miles an hour?

Ans. 6 hours. The meaning is, that the bodies will be together at the end of six hours from the present moment.

(b) What does the result become when $d = 24$ miles, $r = 8$ miles an hour, and $r' = 11$ miles an hour? When $r' = 10$ miles an hour? When $r' = 9$ miles an hour? How is the result affected by diminishing the difference between the two rates?

(c) What does the result become when $d = 5$ miles, $r = 8$ miles an hour, and $r' = 8$ miles an hour?

Ans. $\frac{5}{0}$. It is plain that if two bodies are five miles apart, and both moving the same way at the same rate, no lapse of time would be great enough for one to overtake the other. The quantity denoted by $\frac{5}{0}$ is a quantity greater than any which can be thought of; that is, infinitely great.

223. The symbol ∞ is used to represent ***an infinitely great quantity.***

To gain a clear notion of the nature of the quantity represented by the symbol ∞, let us suppose the rates r and r' to differ by a small amount, say 0.1 (that is, $r = 8$ and $r' = 8.1$), then by a smaller amount, 0.01, then by 0.001, then by 0.000001, and so on, finally diminishing the amount of difference to 0.

Now
$$\frac{5}{0.1} = 50$$
$$\frac{5}{0.01} = 500$$
$$\frac{5}{0.001} = 5000$$
$$\frac{5}{0.000001} = 5000000$$

. . . .

and, finally,
$$\frac{5}{0} = \infty.$$

Here the quotient increases as the divisor diminishes, and in the same proportion. When, finally, the divisor becomes less than any fraction, however small, the quotient becomes larger than any number, however great; that is, infinitely great.

224. The symbol 0 is used to represent ***an infinitely small quantity.***

A quantity which is neither infinitely great nor infinitely small is called a ***finite quantity.***

In the illustration above given, any number a might be used instead of 5. Hence, in general,

$$\frac{a}{0} = \infty$$

or, in words, *a finite quantity divided by an infinitely small quantity gives an infinitely great quotient.*

By reversing the illustration, making the divisor greater and greater, instead of smaller, we may show that

$$\frac{a}{\infty} = 0,$$

or, in words, *a finite quantity divided by an infinitely great quantity gives an infinitely small quotient.*

(d) What does the result become when $d = 24$ miles, $r = 8$ miles an hour, and $r' = 6$ miles an hour?

Ans. -12 hours. The meaning is, that the bodies *were* together twelve hours ago. B is losing, instead of gaining, and the two bodies are getting farther apart.

Although, in the general solution, we assumed that B *would* overtake A in the *future*, yet we see that the general result, rightly interpreted, includes the case in which B *was* overtaken by A in the *past*.

(e) What does the result become when $d = 0$, $r = 8$ miles an hour, and $r' = 8$ miles an hour?

Ans. $\frac{0}{0}$. It is plain that the two bodies are together now, will be together at any time we choose to assign in the future, and were together at any time we choose to assign in the past. The answer cannot be determined: it is an indeterminate quantity.

225. The symbol $\frac{0}{0}$ is used to represent an ***indeterminate quantity.***

Its real value may be a positive quantity, or a negative quantity, or zero, or infinite, as in the case (e) above stated.

Show that $0 \times \infty$ and $\frac{\infty}{\infty}$ are also symbols of indeterminate quantities.

226. To complete the discussion of this problem of pursuit, we add a few more special cases which come under the general result by giving it the proper interpretation.

(f) What does the result become when $d = 24$ miles, $r = 0$, and $r' = 8$ miles an hour?

Ans. $+3$ hours. A is not moving, and B will come to the point where A is three hours hence.

(g) What does the result become when $d = 24$ miles, $r = 8$ miles an hour, and $r' = 0$?

Ans. -3 hours. B is not moving, and A passed the point where B is three hours ago.

(h) What does the result become when $d = 24$ miles, $r = -5$ miles an hour, and $r' = 7$ miles an hour?

Ans. $+2$ hours. The motion of A is reversed. The two bodies are moving towards each other, and will meet two hours hence.

(i) What does the result become when $d = 24$ miles, $r = 5$ miles an hour, and $r' = -7$ miles an hour?

Ans. -2 hours. The motion of B is reversed. The two bodies are moving away from each other, and were together two hours ago.

227. Exercises.

37. Give such numerical values to a, r, and r', in Art. 220, as will make the first answer infinite, and interpret the result.

38. Give such numerical values to a, r, and r', in Art. 220, as will make the first answer indeterminate, and interpret the result.

39. If we suppose the hour-hand of a clock to move *backwards*, and the minute-hand *forwards*, at the usual rates, what numerical values must be given to r and r' in Art. 220? Now, if the hands are together at 12 o'clock, find, by substituting in the proper expressions, when and how often they will be together in the next six hours.

40. A grocer has two sorts of tea, one worth a cents and the other worth b cents a pound. How many pounds of each sort must he take to make a mixture of m pounds, worth c cents a pound?

41. What would the results of the last problem become if the numerical values of a and c were equal, while those of a and b were unequal, a being greater than b? What would the numerical results signify?

42. What would the results become if the numerical values of a, b, and c were equal, and what would the results signify?

SECTION VII.

SIMULTANEOUS EQUATIONS.

228. PROBLEM. A boy bought 5 apples and 4 oranges for 22 cents, and again at the same rates 2 apples and 3 oranges for 13 cents. What was the price of an apple, and the price of an orange?

SOLUTION. Let $x =$ the price of an apple, and $y =$ the price of an orange. Then, by the conditions of the problem,

$$(1)\quad 5x + 4y = 22$$

$$(2)\quad 2x + 3y = 13.$$

Here are two equations with two unknown quantities, from which we must derive one equation with only one unknown quantity. Multiply each member of (1) by 3, and each member of (2) by 4, to make the co-efficient of y the same in both equations:

$$(3)\quad 15x + 12y = 66$$

$$(4)\quad 8x + 12y = 52.$$

Subtract each member of (4) from the corresponding member of (3), and y disappears from the result:

$$(5)\quad 7x = 14$$

whence $x = 2.$

Substitute 2 for x in one of the original equations, say in (1):

$$10 + 4y = 22$$

$$4y = 12$$

$$y = 3.$$

Ans. An apple cost 2 cents, and an orange 3 cents.

229. We thus see that a problem requiring the values of two unknown quantities can be solved by the use of two equations.

The equations must express distinct and independent conditions of the problem, that is, conditions which cannot be inferred one from the other.

To illustrate this, suppose the conditions of the above problem had been that the boy bought 5 apples and 4 oranges for 22 cents, also 10 apples and 8 oranges for 44 cents. These conditions are not independent, for the last may be inferred from the first. The equations

$$5x + 4y = 22$$
$$10x + 8y = 44$$

express, not two distinct conditions, but only one. If either equation is given, the other may be derived from it by multiplying or dividing each member by 2. One of these equations therefore *depends* on the other. The values of the unknown quantities cannot be found from such equations.

230. Equations which express independent conditions of a problem are called ***independent equations.***

To see whether given equations are independent, we find whether one of them can be derived from the other.

In Article 228, are (1) and (2) independent? Are (1) and (3)? Are (2) and (4)? Are (3) and (4)?

231. When two or more equations express conditions of the same problem, they are called ***simultaneous equations.***

In a set of simultaneous equations each symbol of an unknown quantity has the *same meaning and value in all the equations.*

232. The process of deriving from two simultaneous equations a single equation with fewer unknown quantities in it than either original equation had is called ***elimination.***

233. Elimination by Addition or Subtraction.

An example of elimination by subtraction was wrought in Art. 228. Had the signs of the co-efficients of y, the quantity to be eliminated, been unlike, the elimination would have been effected by addition instead of subtraction. The following examples will further illustrate this method.

ILLUSTRATIVE EXAMPLES. I. Given $7x-3y=9$ and $4x+5y=32$, to find the values of x and y.

Multiply both members of (1) by 5, and of (2) by 3, and *add* the results (3) and (4) to get (5).

$$\begin{array}{lrcl}
(1) & 7x-\ \ 3y & = & 9\\
(2) & 4x+\ \ 5y & = & 32\\
(3) & 35x-15y & = & 45\\
(4) & 12x+15y & = & 96\\
(5) & 47x & = & 141\\
 & x & = & 3
\end{array}$$

NOTE. For the sake of brevity we speak of adding or subtracting one equation to or from another; but the meaning is that each member of one equation is added to or subtracted from the corresponding member of the other.

Substitute 3 for x in (1), and find the value of y.

$$\begin{aligned}21-3y&=9\\ -3y&=-12\\ y&=4.\end{aligned}$$

Verify by substituting 3 for x and 4 for y in (2).

$$12+20=32.$$

If we had chosen to eliminate x, the work would have been as follows:

Multiply both members of (1) by 4, and of (2) by 7, and *subtract* the first result (3′) from the second (4′).

$$\begin{array}{lrcl}
(3') & 28x-12y & = & 36\\
(4') & 28x+35y & = & 224\\
(5') & 47y & = & 188\\
 & y & = & 4
\end{array}$$

Substitute 4 for y in (1), and find the value of x.

$$\begin{aligned}7x-12&=9\\ 7x&=21\\ x&=3.\end{aligned}$$

II. Given $\dfrac{x+1}{y}=\dfrac{1}{3}$ and $\dfrac{x}{y+1}=\dfrac{1}{4}$, to find x and y.

Equations (1) and (2) cleared of fractions give (3) and (4). By subtracting 3 and y from each member of (3), and y from each member of (4), the equations are further reduced to the forms (5) and (6). Then y is eliminated by subtracting (5) from (6). The value of y is found from (3).

$$\begin{array}{lrcl}
(1) & \dfrac{x+1}{y} & = & \dfrac{1}{3}\\[2ex]
(2) & \dfrac{x}{y+1} & = & \dfrac{1}{4}\\[2ex]
(3) & 3x+3 & = & y\\
(4) & 4x & = & y+1\\
(5) & 3x-y & = & -3\\
(6) & 4x-y & = & 1\\
(7) & x & = & 4\\
 & 12+3 & = & y\\
 & y & = & 15.
\end{array}$$

Before eliminating by addition or subtraction, equations should always be reduced to simple forms like (5) and (6).

234. From the foregoing illustrations may be derived this statement of the method of elimination by addition or subtraction:

The two equations are first reduced to simple forms with unknown terms only in the first members and known terms in the second members. Then the members of each equation are multiplied by such numbers as will make the co-efficients of the quantity to be eliminated arithmetically equal in the two results. Finally, if these co-efficients have unlike signs, the results are added, but, if the co-efficients have like signs, one result is subtracted from the other.

235. Elimination by Comparison.

ILLUSTRATIVE EXAMPLE. Given $5x+4y=22$ and $2x+3y=13$, to find the values of x and y.

From each member of (1) subtract $5x$, and divide each member of the result by 4. This gives (3). In the same way (4) is derived from (2). Equation (5) is derived from (3) and (4) by applying the principle that quantities equal to the same quantity are equal to each other. From (5) the value of x is found.

Substitute 2 for x in (1) to find y.

We might have begun by eliminating x, using a like process. Let the example be wrought in this way as an exercise.

$$\begin{aligned}
&(1) & 5x+4y&=22\\
&(2) & 2x+3y&=13\\
&(3) & y&=\frac{22-5x}{4}\\
&(4) & y&=\frac{13-2x}{3}\\
&(5) & \frac{13-2x}{3}&=\frac{22-5x}{4}\\
& & 52-8x&=66-15x\\
& & 7x&=14\\
& & x&=2\\
& & 10+4y&=22\\
& & 4y&=12\\
& & y&=3.
\end{aligned}$$

236. From this example may be derived the following statement of the method of elimination by comparison;

From each equation is derived an expression for the value of the quantity to be eliminated, and with these two expressions a new equation is formed on the principle that two quantities each equal to a third quantity are equal to each other. (Axiom 7.)

237. Elimination by Substitution.

ILLUSTRATIVE EXAMPLE. Given $5x + 4y = 22$ and $2x + 3y = 13$, to find the values of x and y.

From (1) is derived (3), an expression for the value of y.

This expression is substituted for y in (2), giving (4).

From (4) is found the value of x, which, substituted in (2), gives the value of y.

We might have begun by eliminating x, using a like process. Let the example be wrought in this way as an exercise.

$$(1)\qquad 5x + 4y = 22$$

$$(2)\qquad 2x + 3y = 13$$

$$(3)\qquad y = \frac{22 - 5x}{4}$$

$$(4)\qquad 2x + \frac{3(22 - 5x)}{4} = 13$$

$$8x + 66 - 15x = 52$$

$$-7x = -14$$

$$x = 2.$$

$$4 + 3y = 13$$

$$3y = 9$$

$$y = 3.$$

238. From this example may be derived the following statement of the method of elimination by substitution:

From one equation is derived an expression for the value of the quantity to be eliminated, and this expression is substituted for the symbol of that quantity in the other equation.

239. Exercises in Elimination.

In general, the method of elimination by addition or subtraction is the most convenient, since it does not introduce fractions. There are, however, many instances in which one of the other methods would make the work simpler. To qualify himself to make a wise choice of method, the learner should frequently work the same exercise by all three methods, and compare the work.

1. $\begin{cases} 3x+5y=8 \\ 4x+3y=7. \end{cases}$

2. $\begin{cases} 3x-5y=13 \\ 2x+7y=81. \end{cases}$

3. $\begin{cases} 54x-121y=15 \\ 36x-77y=21. \end{cases}$

4. $\begin{cases} 45x+8y=350 \\ 21y-13x=132. \end{cases}$

5. $\begin{cases} 9x-4y=8 \\ 13x+7y=101. \end{cases}$

6. $\begin{cases} 101x-24y=63 \\ 103x-28y=29. \end{cases}$

7. $\begin{cases} 2x+3y=8. \\ 3x+7y=7. \end{cases}$

8. $\begin{cases} 7y-3x=139. \\ 2x+5y=91. \end{cases}$

9. $\begin{cases} 4x+9y=106. \\ 8x+17y=198. \end{cases}$

10. $\begin{cases} 17x+12y=59. \\ 19x-4y=153. \end{cases}$

11. $\begin{cases} 69y-17x=103. \\ 14x-13y=-41. \end{cases}$

12. $\begin{cases} 2x-7y=8. \\ 4y-9x=19. \end{cases}$

13. $\begin{cases} x:y=4:5 \\ x+y=81. \end{cases}$

14. $\begin{cases} 2\frac{1}{4}x+3\frac{1}{3}y=48 \\ 4\frac{1}{2}x+10y=126. \end{cases}$

15. $\begin{cases} \frac{1}{9}x+\frac{1}{8}y=43. \\ \frac{1}{8}x+\frac{1}{9}y=42. \end{cases}$

16. $\begin{cases} 3\frac{1}{2}x-4\frac{1}{2}y=12 \\ 7x+9y=60. \end{cases}$

17. $\begin{cases} 2.25x+3y=45 \\ 4.5x+10y=216. \end{cases}$

18. $\begin{cases} 4.25x-3.5y=6 \\ 0.5x-0.25y=2. \end{cases}$

19. $\begin{cases} x+y=s \\ x-y=d. \end{cases}$

20. $\begin{cases} x:y=a:b \\ x+y=s. \end{cases}$

21. $\begin{cases} x+y=1 \\ \dfrac{x}{a}+\dfrac{y}{b}=\dfrac{1}{c}. \end{cases}$

22. $\begin{cases} ax+by=m \\ a'x+b'y=m'. \end{cases}$

23. $$\begin{cases}\frac{6}{x}+\frac{5}{y}=4\\ \frac{8}{x}-\frac{15}{y}=1.\end{cases}$$

24. $$\begin{cases}\frac{1}{x}+\frac{1}{y}=\frac{1}{3}\\ \frac{1}{x}-\frac{1}{y}=\frac{1}{6}.\end{cases}$$

25. $$\begin{cases}\frac{1}{x}+\frac{2}{y}=\frac{11}{15}\\ \frac{3}{x}+\frac{4}{y}=1\frac{4}{5}.\end{cases}$$

26. $$\begin{cases}\frac{a}{x}+\frac{b}{y}=m\\ \frac{a'}{x}+\frac{b'}{y}=m'.\end{cases}$$

27. $$\begin{cases}\frac{x+2}{7}+\frac{y-x}{4}=2x-8\\ \frac{2y-3x}{3}+2y=3x+4.\end{cases}$$

28. $$\begin{cases}\frac{3x-5y}{2}+3=\frac{2x+y}{5}\\ 8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}.\end{cases}$$

29. $$\begin{cases}\frac{x+11}{10}+\frac{y-4}{6}=x-7\\ \frac{x+5}{7}-\frac{y-7}{3}=3y-x.\end{cases}$$

30. $$\begin{cases}2x-\frac{y-3}{5}=\frac{5x-2}{2}\\ 2y-\frac{x-5}{3}=\frac{7y-7}{2}.\end{cases}$$

31. $$\begin{cases}\frac{2x+y}{9}+\frac{7y+6x+11}{18}=\frac{19}{2}-\frac{5x-17}{6}\\ \frac{3}{7}(5x+3y+2)=\frac{1}{2}(9y+6).\end{cases}$$

32. $$\begin{cases}\frac{3x-5y}{3}-\frac{2x-8y-9}{12}=\frac{y}{2}+\frac{1}{3}+\frac{1}{4}\\ 3\frac{1}{2}\left(\frac{x}{7}+\frac{y}{4}+1\frac{1}{3}\right)=3\frac{1}{3}\left(4x-\frac{y}{8}-24\right).\end{cases}$$

33. $$\begin{cases}x+y=38\frac{1}{5}-\frac{4x+12y}{10}\\ \frac{1}{21x}+\frac{1}{22y}=\frac{452}{462xy}.\end{cases}$$

34. $$\begin{cases}2.4x+0.32y-\frac{0.36x-0.05}{0.5}=0.8x+\frac{2.6+0.005y}{0.25}\\ \frac{0.04y+0.1}{0.3}=\frac{0.07x-0.1}{0.6}.\end{cases}$$

35. $$\begin{cases} (x+5)(y+7)=(x+1)(y-9)+112 \\ 2x+10=3y+1. \end{cases}$$

36. $$\begin{cases} 16x+6y-1=\dfrac{128x^2-18y^2+217}{8x-3y+2} \\ \dfrac{10x+10y-35}{2x+2y+3}=5-\dfrac{54}{3x+2y-1}. \end{cases}$$

240. Problems to be solved by the Use of two Equations with two Unknown Quantities.

37. A drover sells to one man 5 cows and 7 oxen for \$370, and to another, at the same prices, 10 cows and 3 oxen for \$355. Required the price of a cow and that of an ox.

38. A grocer has two grades of sugar. If he mixes 3 pounds of the poorer with 5 pounds of the better, the mixture will be worth $10\frac{1}{4}$ cents a pound; but, if he mixes 5 pounds of the poorer with 3 pounds of the better, the mixture will be worth $9\frac{3}{4}$ cents a pound. What is the worth of each grade?

39. If you can get 12 marbles by taking half of John's and a third of Henry's, or 13 by taking a third of John's and half of Henry's, how many marbles has each?

40. A tank containing 192 gallons is provided with two faucets. Both faucets are opened for three minutes, when one of them is shut, and the tank is emptied by the other in eleven minutes more. Had six minutes elapsed before the first faucet was shut, the tank would have been emptied by the other in six minutes more. How many gallons are discharged by each faucet in a minute?

41. A market-woman bought hen's eggs at the rate of 2 for 3 cents, and duck's eggs at the rate of 3 for 2 cents, paying 95 cents for all. She afterwards sold them for \$1.35, thereby gaining half a cent on each egg. How many of each kind did she buy?

42. A mule and an ass were carrying loads of wheat to mill. The ass, complaining of his load, said to the mule,

"I need only a hundred-weight of your load to make mine twice as heavy as yours." To which the mule replied, "But, if you give me a hundred-weight of yours, my load will then be three times as heavy as yours." How many hundred-weight did each carry?

43. Find two numbers such that one shall be as much greater than 10 as the other is less than 10, and one-tenth of their sum equal to one-fourth of their difference.

44. Seven years ago a father was four times as old as his son; but seven years hence he will be only twice as old. What is the age of each?

45. Find two numbers whose sum is 224, and whose ratio is that of 15 to 17.

46. Find two numbers whose sum is a, and whose ratio is that of m to n.

47. Two pounds of tea and three pounds of sugar cost $1.44; but, if tea were to rise 40 % and sugar 25 %, the cost would be $1.98. What are the present prices?

48. The income from a certain farm exceeds the expenses by $2000; but, if the income were diminished 20 % and the expenses diminished 25 %, the excess of income over expenses would be $1800. What are the income and expenses at present?

49. A grocer sold 50 pounds of tea at an advance of 10 % on the cost, and 30 pounds of coffee at an advance of 20 % on the cost. He received for the whole $27.40, thereby gaining $2.90. What was the cost per pound of the tea and of the coffee?

50. There is a fraction which becomes equal to $\frac{1}{2}$ when both numerator and denominator are diminished by 1, and which becomes equal to $\frac{1}{3}$ when 2 is taken from the numerator and added to the denominator. What is the fraction?

51. If both numerator and denominator of a certain fraction be increased by 2, its value will be $\frac{3}{5}$; but, if both be increased by 3, its value will be $\frac{5}{8}$. What is the fraction?

52. Find two numbers whose ratio is that of a to b, but,

if k is added to each of them, the ratio of the results will be that of a' to b'.

53. A tank is supplied by two pumps. One day the tank was half filled by working both pumps three hours, and on the next day it was two-thirds filled by working the larger pump two hours and the smaller seven hours. In how many hours would either pump worked alone fill the tank?

54. The dimensions of a rectangular piece of land are such, that, if it were 2 rods wider and 3 rods longer, its area would be 64 square rods greater; but, if it were 3 rods wider and 2 rods longer, its area would be 68 square rods greater. What are the dimensions?

55. A sailing-party having hired a boat for a certain sum found, that, if their number had been four more, each would have had 25 cents less to pay, but, if their number had been three less, each would have had 25 cents more to pay. What was their number, and what was paid by each?

56. A certain sum of money at simple interest amounts to $1375 in 15 months, and to $1425 in 21 months. Find the principal and the rate of interest.

57. A certain sum of money at simple interest will amount to a dollars in m months and to b dollars in n months. Find the principal and the rate of interest.

58. Two boys, Arthur and Philip, run a race of a mile, making two trials. In the first, Arthur gives Philip a start of 200 feet, and beats him by 35 seconds. In the second, Arthur gives Philip a start of 1 minute and 10 seconds, and gets beaten by 88 feet. How long does it take each of them to run a mile?

59. A brewer has two casks with a certain quantity of beer in each. He draws out of the first into the second as much as there was in the second to begin with; then he draws out of the second into the first as much as was left in the first; and then, again, out of the first into the second as much as was left in the second. There are now 24 gallons in each cask. How much was there in each cask at first?

60. A student walking in from Cambridge to Boston notices that horse-cars coming out meet him every 3 minutes, and those going in overtake him every 9 minutes. Supposing the cars to be started at regular intervals, so that each car is half a mile ahead of the next one, how many miles an hour is the student walking, and how fast do the horse-cars go?

61. Hiero, king of Syracuse, is said to have given his goldsmith twenty pounds of gold with which to make a crown. When finished, the crown weighed twenty pounds; but the king, suspecting that it had been alloyed with silver, consulted Archimedes. This philosopher detected the fraud by plunging the crown in water. Knowing that pure gold would displace $\frac{5}{97}$ of its weight, and pure silver $\frac{2}{21}$ of its weight of water, he observed that the crown displaced $\frac{1}{16}$ of its weight of water. From these data he could compute the quantity of gold and the quantity of silver in the crown, on the supposition that no other metal was used in the alloy. What should have been his results?

Specific Gravity.

The specific gravity of a body is the ratio of its weight to the weight of an equal bulk of pure water.

$$\frac{\text{Weight of a body}}{\text{Weight of an equal bulk of water}} = \text{Specific gravity.}$$

A body when immersed in a liquid is buoyed up by a force equal to the weight of the liquid displaced. When, therefore, a body is weighed first in the air and then under water, there is an apparent loss of weight, and this loss of weight is equal to the weight of the water displaced. Hence

$$\frac{\text{Weight of a body in air}}{\text{Loss of weight in water}} = \text{Specific gravity.}$$

From these equations may be derived the formula

$$\begin{aligned}\frac{\text{Weight of a body}}{\text{Its specific gravity}} &= \text{Weight of an equal bulk of water.}\\ &= \text{Loss of weight in water.}\end{aligned}$$

The principle here stated may be applied in the solution of the following problems.

62. When p pounds of a substance having a specific gravity s are mixed with p' pounds of another substance having a specific gravity s', what is the specific gravity of the mixture, supposing that no change of bulk takes place?

63. Two substances of known specific gravities, s and s', are mixed in such proportions that the specific gravity of the mixture is s''. Find the per cent of each substance in the mixture.

64. In one experiment, w pounds of wood and m pounds of metal are tied together, and, when weighed under water, found to lose l pounds. In another experiment, w' pounds of the same kind of wood and m' pounds of the same kind of metal are found to lose l' pounds in water. From these data find the specific gravity of the wood and that of the metal.

65. The specific gravity of pure milk is 1.031. If the specific gravity of some "watered milk" is 1.018, what per cent of the mixture is pure milk, and what per cent water?

66. Two pounds of pine wood and six pounds of cast-iron being tied together and weighed under water are observed to lose 4.4 pounds, while three pounds of the wood and four pounds of the iron lose 5.9 pounds. From these data find the specific gravity of the wood and that of the iron.

67. What per cent of absolute alcohol (specific gravity $=$ 0.795) is there in "proof spirit" (specific gravity $= 0.920$)?

68. The silver coins of the United States are made $\frac{9}{10}$ by weight of silver (specific gravity $= 10.5$) and $\frac{1}{10}$ of copper (specific gravity $= 8.9$), what is the specific gravity of these coins?

Three or more Unknown Quantities.

241. When there are three or more equations with as many unknown quantities, the methods of elimination already explained may be used. The method by addition or subtraction is almost always the most convenient.

ILLUSTRATIVE EXAMPLE. Given

$$\begin{aligned}
&(1)\quad 2x+3y+4z=16\\
&(2)\quad 3x+2y-5z=\ 8\\
&(3)\quad 5x-6y+3z=\ 6
\end{aligned}$$

to find the values of x, y, and z.

First eliminate one of the unknown quantities, say z, from any pair of the equations, say (1) and (2). For this purpose multiply the members of (1) by 5, and those of (2) by 4, and add the results:

$$\begin{aligned}
&(4)\quad 10x+15y+20z=\ 80\\
&(5)\quad 12x+\ 8y-20z=\ 32\\
&(6)\quad 22x+23y\qquad\quad =112.
\end{aligned}$$

Now eliminate the same unknown quantity, z, from another pair of equations, say (1) and (3); for which purpose multiply the members of (1) by 3, and those of (3) by 4, and subtract the latter result from the former:

$$\begin{aligned}
&(7)\quad\ \ 6x+\ 9y+12z=48\\
&(8)\quad\ 20x-24y+12z=24\\
&(9)\quad -14x+33y\qquad\ \ =24.
\end{aligned}$$

We have now two equations (6) and (9) with only two unknown quantities, x and y. From these eliminate either unknown quantity, say x; for which purpose multiply the members of (6) by 7, and those of (9) by 11, and add the results:

$$\begin{aligned}
&(10)\quad\ \ 154x+161y=784\\
&(11)\quad -154x+363y=264\\
&(12)\quad\qquad\qquad 524y=1048\\
&\qquad\qquad\qquad\quad\ \ y=2
\end{aligned}$$

To find x, substitute 2 for y in (6) or (9), say in (6).

$$\begin{aligned}
22x+46&=112\\
22x&=66\\
x&=3.
\end{aligned}$$

To find z, substitute 2 for y and 3 for x in (1), (2), or (3), say in (1).

$$\begin{aligned}
6+6+4z&=16\\
4z&=4\\
z&=1.
\end{aligned}$$

It is easy to see that the elimination might have been effected in other ways. For example, we might have begun by eliminating x, obtaining two equations in terms of y and z; or by eliminating y, obtaining two equations in terms of x and z.

242. The method illustrated by the foregoing example may be used to find the values of any number of unknown quantities from an equal number of equations. If there are *n* unknown quantities to be found, there must be *n* independent equations given to find them. The following is a general statement of the method:

From different pairs of the n *given equations the same unknown quantity is eliminated, until* $n-1$ *equations are obtained with only* $n-1$ *unknown quantities. From these* $n-1$ *equations, by a similar process, are obtained* $n-2$ *equations with only* $n-2$ *unknown quantities. And this is repeated until there results one equation with only one unknown quantity, which being solved gives the value of that unknown quantity. This value being substituted for the unknown quantity in an equation containing it and one other unknown quantity, and the equation being solved, the value of another unknown quantity is found. These two values are used in a similar way to find the value of a third unknown quantity, then three values are used to find a fourth, and so on until all are found.*

NOTE. The work of elimination will be shorter when, as is frequently the case, some one or more of the given equations have less than the full number of unknown quantities.

243. Exercises in Elimination.

69. $\begin{cases} 3x - y + z = 16 \\ 5x + 3y - 2z = 13 \\ 7x + 4y - 5z = 7. \end{cases}$

70. $\begin{cases} 3x + 5y - 4z = 25 \\ 5x - 2y + 3z = 46 \\ -x + 3y + 5z = 62. \end{cases}$

71. $\begin{cases} x + y - z - 8 = 0 \\ x - y + z - 9 = 0 \\ -x + y + z - 10 = 0. \end{cases}$

72. $\begin{cases} 3x + 4z = 57 \\ 5x + 3y = 5 \\ 2y + z = -11. \end{cases}$

73. $\begin{cases} 2x+4y+3z-22=0 \\ 4x-2y+5z+22=0 \\ 6x+7y-\ \ z-71=0. \end{cases}$

74. $\begin{cases} x+\dfrac{y}{2}=1 \\ y+\dfrac{z}{3}=1 \\ z+\dfrac{x}{4}=1. \end{cases}$

75. $\begin{cases} x+\dfrac{y+z}{2}=102 \\ y+\dfrac{z+x}{3}=\ \ 78 \\ z+\dfrac{x+y}{4}=\ \ 61. \end{cases}$

76. $\begin{cases} y+\frac{1}{2}x=41 \\ x+\frac{1}{4}z=20\frac{1}{2} \\ y+\frac{1}{5}z=34. \end{cases}$

77. $\begin{cases} x+y=a \\ y+z=b \\ z+x=c. \end{cases}$

78. $\begin{cases} \dfrac{x}{2}+\dfrac{y}{3}+\dfrac{z}{4}=41 \\ \dfrac{x}{3}+\dfrac{y}{4}+\dfrac{z}{5}=31 \\ \dfrac{x}{4}+\dfrac{y}{5}+\dfrac{z}{6}=25. \end{cases}$

79. $\begin{cases} 4x-3y+2u=\ \ 7 \\ 2x+6z=28 \\ 4u-2y=14 \\ 3x+4u=26. \end{cases}$

80. $\begin{cases} w+x+y=12 \\ w+x+z=15 \\ w+y+z=17 \\ x+y+z=22. \end{cases}$

81. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{a} \\ \dfrac{1}{w}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{b} \\ \dfrac{1}{w}+\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{1}{c} \\ \dfrac{1}{w}+\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{d}. \end{cases}$

82. $\begin{cases} x+2y+\ \ 4z+\ \ \ 8=0 \\ x+3y+\ \ 9z+\ \ 27=0 \\ x+5y+25z+125=0. \end{cases}$

83. $\begin{cases} x+ay+a^2z+a^3=0 \\ x+by+b^2z+b^3=0 \\ x+cy+c^2z+c^3=0. \end{cases}$

84. $\begin{cases} v+2x+\ \ 4y+\ \ \ 8z=\ \ 26 \\ v+3x+\ \ 9y+\ \ 27z=\ \ 58 \\ v+5x+25y+125z=194 \\ v+7x+49y+343z=466. \end{cases}$

85. $\begin{cases} v-2w+\ \ 2x+4y+\ \ 6z=41 \\ -v+2w-\ \ 4x+5y-\ \ 2z=\ \ 4 \\ 4v-3w+\ \ 6x+2y+\ \ 4z=43 \\ 3v+5w+\ \ 8x-3y+10z=62 \\ 5v+4w+10x+2y-\ \ 8z=28. \end{cases}$

244. Problems to be solved by the Use of Three or more Equations with Three or more Unknown Quantities.

86. At a fruit-stand, one girl bought 3 peaches, 2 pears, and 8 apples for 20 cents; another bought 4 peaches, 5 pears, and 6 apples for 29 cents; and another bought 5 peaches, 7 pears, and 10 apples for 41 cents. What was the price of a peach, of a pear, and of an apple?

87. Find three numbers such that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third shall be equal to 62; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third, equal to 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third, equal to 38.

88. Three boys, A, B, and C, hired a boat for $5, and one of them was to go and pay for it; but neither alone had money enough. A said to B, "My money with a half of yours will be enough." B replied, "My money with only a third of yours will be enough." C said to A, "Take a fourth of my money and put it with yours, and you will have enough." How much money had each?

89. A and B together can reap a field of rye in $3\frac{3}{4}$ days, B and C could reap it in $4\frac{2}{7}$ days, but it would take C and A 6 days. Find the time in which either alone could reap the field, and the time in which all working together could do it.

90. A and B together can do a piece of work in l days, B and C in m days, C and A in n days. In how many days could each person alone do it?

91. A certain number is expressed by three figures. The digit-value of the middle figure is equal to half the sum of the digit-values of the other two. If the number be divided by the sum of the digit-values of its three figures, the quotient will be 48. Moreover, if the number be diminished by 198, the remainder will be expressed by the same three figures written in the reverse order. What is the number?

HINT. Let x, y, and z represent the digit-values of the hundreds', tens', and units' figures respectively. Then the number itself will be represented by $100x + 10y + z$.

92. There are three ingots composed of different metals in various proportions. The first contains 42 % copper, 18 % zinc, and 40 % tin; the second, 75 % copper, 18 % zinc, and 7 % tin; and the third, 25 % copper, 45 % zinc, and 30 % tin. How much must be taken from each of these to form a new ingot containing 25 pounds of copper, 10 of zinc, and 15 of tin?

93. A grocer has four sorts of tea. If he mixes 2 lbs. of the first, 3 lbs. of the second, 7 lbs. of the third, and 4 lbs. of the fourth, the mixture will be worth $46\frac{7}{8}$ cents a pound. If he mixes 3 lbs. of the first, 5 lbs. of the second, 6 lbs. of the third, and 2 lbs. of the fourth, the mixture will be worth $44\frac{1}{16}$ cents a pound. If he mixes 5 lbs. of the first, 4 lbs. of the second, 3 lbs. of the third, and 4 lbs. of the fourth, the mixture will be worth $43\frac{15}{16}$ cents a pound. If he mixes 4 lbs. of the first, 2 lbs. of the second, 2 lbs. of the third, and 8 lbs. of the fourth, the mixture will be worth $47\frac{3}{4}$ cents a pound. What is the value of a pound of each sort?

94. Four gold coins are in a purse. If one of them is taken out, the remaining three are worth $35, or $31, or $26, or $16, according to which of the four is taken out. What are the values of these coins?

95. Four men, A, B, C, and D, had engaged to do a piece of work; but one of them was called away. If A went, the rest could do the work in 9 days; or, if B went, the rest could do it in 10 days; or, if C went, the rest could do it in 12 days; or, if D went, the rest could do it in 15 days. In what time could each do it alone? all together?

96. Divide the number 192 into four parts such that if the first is increased by 7, the second diminished by 7, the third multiplied by 7, and the fourth divided by 7, the sum, remainder, product, and quotient shall all be equal to one another.

97. Divide the number *a* into four parts such that if the first is increased by *b*, the second diminished by *b*, the third multiplied by *b*, and the fourth divided by *b*, the sum, remainder, product, and quotient shall all be equal to one another.

SECTION VIII.

POWERS AND ROOTS.

245. A power, being a product of equal factors, may always be found by actual multiplication. In the case of numbers this is the only way.

In the case of monomials with literal factors, the method is *to raise the numerical co-efficient of the monomial to the required power, and then multiply the exponent of each literal factor by the exponent of the required power.*

This may be expressed briefly by the formula

$$(A\,x^p\,y^q\,\ldots)^m = A^m\,x^{mp}\,y^{mq}\,\ldots$$

where A stands for the numerical co-efficient, and x^p, y^q, . . . for the literal factors.

See Article 82, Exercises 100 to 117 and 135 to 139.

The Binomial Theorem.

246. In the case of binomials, the labor of actual multiplication may be saved by using the binomial theorem. A formula expressing this theorem may be derived from the following results obtained by actual multiplication:

$$(a+b)^1 = a+b.$$
$$(a+b)^2 = a^2+2\,ab+b^2.$$
$$(a+b)^3 = a^3+3\,a^2b+3\,ab^2+b^3.$$
$$(a+b)^4 = a^4+4\,a^3b+6\,a^2b^2+4\,ab^3+b^4.$$
$$(a+b)^5 = a^5+5\,a^4b+10\,a^3b^2+10\,a^2b^3+5\,ab^4+b^5.$$
$$(a+b)^6 = a^6+6\,a^5b+15\,a^4b^2+20\,a^3b^3+15\,a^2b^4+6ab^5+b^6.$$

247. An examination of these results will show:

1. *That the leading quantity,* a, *enters into every term of the development except the last; that its exponent in the first term is equal to the exponent of the power, and is diminished by* 1 *in each successive term towards the right, to the last term, where it vanishes.*

2. *That the following quantity,* b, *enters into every term of the development except the first; that its exponent in the second term is* 1, *and is increased by* 1 *in each successive term towards the right to the last term, where it is equal to the exponent of the power.*

3. *That the co-efficient of the first term is* 1; *and the co-efficient of any term after the first may be found by multiplying the co-efficient of the preceding term by the exponent of the leading quantity in that term, and dividing the product by the number of terms that precede the required term.*

248. These statements are embodied in the formula

$$(a+b)^m = a^m + \frac{m}{1}a^{m-1}b + \frac{m(m-1)}{1\times 2}a^{m-2}b^2 + \frac{m(m-1)(m-2)}{1\times 2\times 3}a^{m-3}b^3 + \dots$$

This formula is true for the first six powers of $a+b$, as may be shown by substituting 1, 2, 3, 4, 5, or 6 for m, reducing the results, and comparing them with the results obtained by actual multiplication; but we have yet to prove that it is true beyond the sixth power, *true, in fact, for all positive integral values of* m.

249. As a preliminary step we shall prove the following proposition:

If the formula given above is true for any one positive integral value of m, *it is also true for the next higher integral value of* m.

PROOF. Let k be a positive integral value of m for which the formula is true. We are to prove that the formula is also true when the value of m is $k+1$.

Substituting k for m in the formula, we get the equation,

$$(a+b)^k=a^k+\frac{k}{1}a^{k-1}b+\frac{k(k-1)}{1\times 2}a^{k-2}b^2+\frac{k(k-1)(k-2)}{1\times 2\times 3}a^{k-3}b^3+\dots$$

which is allowed to be true.

If we multiply each member of this equation by $a+b$, the first member becomes $(a+b)^{k+1}$.

To find the second member, we multiply as follows:

$$\begin{array}{llll}
a^k+\frac{k}{1}a^{k-1}b+ & \frac{k(k-1)}{1\times 2}a^{k-2}b^2+ & \frac{k(k-1)(k-2)}{1\times 2\times 3}a^{k-3}b^3+\dots & \\
a+b & & & \\
\hline
a^{k+1}+\frac{k}{1}a^k b+ & \frac{k(k-1)}{1\times 2}a^{k-1}b^2+ & \frac{k(k-1)(k-2)}{1\times 2\times 3}a^{k-2}b^3+\dots & \\
\qquad a^k b+ & \frac{k}{1}a^{k-1}b^2+ & \frac{k(k-1)}{1\times 2}a^{k-2}b^3+\dots & \\
\hline
a^{k+1}+\frac{k+1}{1}a^k b+ & \frac{(k+1)k}{1\times 2}a^{k-1}b^2+ & \frac{(k+1)k(k-1)}{1\times 2\times 3}a^{k-2}b^3+\dots &
\end{array}$$

In adding the partial products, the co-efficients of like terms are reduced to a common denominator, and united, as follows:

The co-efficient of the second term,

$$\frac{k}{1}+1=\frac{k+1}{1},$$

the co-efficient of the third term,

$$\frac{k(k-1)}{1\times 2}+\frac{k}{1}=\left[\frac{k-1}{1\times 2}+\frac{1}{1}\right]k=\frac{(k+1)k}{1\times 2},$$

the co-efficient of the fourth term,

$$\frac{k(k-1)(k-2)}{1\times 2\times 3}+\frac{k(k-1)}{1\times 2}=\left[\frac{k-2}{1\times 2\times 3}+\frac{1}{1\times 2}\right]k(k-1)=\frac{(k+1)k(k-1)}{1\times 2\times 3},$$

and so on.

The result, therefore, of multiplying each member of the above equation by $a+b$ is,

$$(a+b)^{k+1}=a^{k+1}+\frac{k+1}{1}a^k b+\frac{(k+1)k}{1\times 2}a^{k-1}b^2+\frac{(k+1)k(k-1)}{1\times 2\times 3}a^{k-2}b^3+,$$

which is true, since it came from multiplying equal quantities by the same quantity.

But precisely this result comes from the formula when $k+1$ is substituted for m. Hence the formula is true when m has the value $k+1$, *which was to be proved.*

250. Now, the formula has been shown to be true when the value of m is 6; then, by reason of the proposition just proved, it must also be true when the value of m is 7. Being true when $m = 7$, it must, for the same reason, be true when $m = 8$; and so on in succession for all positive integral values of m.

Hence the formula,

$$(a+b)^m = a^m + \frac{m}{1}a^{m-1}b + \frac{m(m-1)}{1\times 2}a^{m-2}b^2 + \frac{m(m-1)(m-2)}{1\times 2\times 3}a^{m-3}b^3 + \dots$$

is true for all positive integral values of m.

This formula expresses in algebraic language the Binomial Theorem of Sir Isaac Newton.

NOTE. The binomial theorem is also true for negative and fractional values of the exponent; but it is beyond the scope of this book to prove that, or to use the formula for such values of the exponent.

251. If $-b$ is substituted for $+b$ in the binomial formula, the result is,

$$(a-b)^m = a^m - \frac{m}{1}a^{m-1}b + \frac{m(m-1)}{1\times 2}a^{m-2}b^2 - \frac{m(m-1)(m-2)}{1\times 2\times 3}a^{m-3}b^3 + \dots$$

Every term containing an odd power of $-b$ becomes negative. These are the 2d, 4th, 6th, &c. In all other particulars the formula remains the same.

252. If we apply the binomial formula to write out the m^{th} power of $b + a$, the result is,

$$(b+a)^m = b^m + \frac{m}{1}b^{m-1}a + \frac{m(m-1)}{1\times 2}b^{m-2}a^2 + \frac{m(m-1)(m-2)}{1\times 2\times 3}b^{m-3}a^3 + \dots$$

The second member of this equation is the same as the second member of the binomial formula, except that the order of terms is reversed. By comparing the corresponding terms of the two we see that,

When the second member of the binomial formula is written out in full, terms equally removed from the extreme terms have equal co-efficients.

253. **Illustrative Example.** Find the 6th power of $3l^2x - \frac{1}{2}y$.

Put $3l^2x = a$, and $\frac{1}{2}y = b$. Then $(3l^2x - \frac{1}{2}y)^6 = (a-b)^6$.

By the binomial formula,

$$(a-b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6.$$

Substituting in this,

for a,	$3l^2x$,	and for b,	$\frac{1}{2}y$,
a^2,	$9l^4x^2$,	b^2,	$\frac{1}{4}y^2$,
a^3,	$27l^6x^3$,	b^3,	$\frac{1}{8}y^3$,
a^4,	$81l^8x^4$,	b^4,	$\frac{1}{16}y^4$,
a^5,	$243l^{10}x^5$,	b^5,	$\frac{1}{32}y^5$,
a^6,	$729l^{12}x^6$,	b^6,	$\frac{1}{64}y^6$,

and reducing each term to its simplest form, the result is,

$$(3l^2x - \tfrac{1}{2}y)^6 = 729l^{12}x^6 - 729l^{10}x^5y + \tfrac{1215}{4}l^8x^4y^2 - \tfrac{135}{2}l^6x^3y^3 + \tfrac{135}{16}l^4x^2y^4 - \tfrac{9}{16}l^2xy^5 + \tfrac{1}{64}y^6.$$

254. Exercises.

1. Expand $(x+y)^9$.
2. Expand $(m-n)^7$.
3. Expand $(x-1)^8$.
4. Expand $(a-1)^{10}$.
5. Expand $(y-2)^7$.
6. Expand $(3-m)^5$.
7. Expand $(a^2-b^3)^4$.
8. Expand $(2x-3y)^4$.
9. Expand $(m^3-n^3)^5$.
10. Expand $(a^m+b^m)^4$.
11. Expand $(1-k^2)^8$.
12. Expand $(2+\frac{1}{2}x)^7$.
13. Expand $(\frac{1}{3}x+\frac{1}{2}y)^5$.
14. Expand $\left(\frac{x}{y}+\frac{y}{x}\right)^6$.
15. Expand $\left(1-\frac{1}{x}\right)^7$.
16. Expand $\left(x+\frac{1}{x}\right)^5$.
17. Expand $(a+2bx)^6$.
18. Expand $(1+5z)^5$.
19. Expand $\left(\frac{1}{m}-\frac{1}{n}\right)^8$.
20. Expand $\left(\frac{2al^2}{m}-\frac{1}{4}m^2y\right)^5$.

255. The powers of a polynomial may be found by actual multiplication; and, in cases of common occurrence, this method is as short as any other.

256. Exercises.

21. Find the fourth power of $a+b+c$.
22. Find the third power of $a+bx+cx^2$.

23. Find the fifth power of $y^2 + 2y + 3$.

24. Find the third power of $a - b + c - d$.

25. Find the third power of $2r^4 + 3r^3 + 4r^2 + 3r + 2$.

26. Show that

$$(a + b + c + d)^2 = a^2 + 2ab + b^2 + 2(a + b)c + c^2 + 2(a + b + c)d + d^2.$$

27. Show that

$$(a + b + c + d + e)^2 = a^2 + 2ab + b^2 + 2(a + b)c + c^2 + 2(a + b + c)d + d^2 + 2(a + b + c + d)e + e^2.$$

28. From the last two exercises, derive a rule for writing down, without multiplication, the square of any polynomial.

29. Apply the rule to expand $(i + j + k + l + m + n)^2$.

30. Apply the rule to expand $(p + q + r + s + t + u + v)^2$.

Roots.

257. One of the equal factors which, multiplied together, produce a quantity is the ***root*** of that quantity.

One of *two* equal factors which produce a quantity is the second root, or square root, of that quantity; one of *three* equal factors is the third root, or cube root; and, in general, one of n equal factors is the n^{th} root.

258. The root of a quantity is indicated by writing before its symbol the sign $\sqrt{}$, which is called the ***radical sign.*** In all cases except that of the square root, a small figure or letter, called the ***index*** of the radical, is written above the radical sign to show what root of the quantity is indicated.

Thus, $\sqrt{a}$ indicates, and is read, "the second root, or square root, of a."

$\sqrt[3]{a}$ indicates, and is read, "the third root, or cube root, of a."

$\sqrt[n]{a}$ indicates, and is read, "the n^{th} root of a."

259. In all cases where a quantity can be recognized as a perfect n^{th} power, the n^{th} root may be found by the simple process of factoring, as already explained.

See Articles 118, 119, 128, 129.

The Square Root of Numbers.

260. The squares of the numbers

1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

are

1, 4, 9, 16, 25, 36, 49, 64, 81, 100;

from which we see that, when the root is expressed by *one* figure, the square is expressed by *one* or *two* figures.

The squares of the numbers

10, 20, 30, 40, 50, 60, 70, 80, 90, 100,

are

100, 400, 900, 1600, 2500, 3600, 4900, 6400, 8100, 10000,

from which we see that, when the root is expressed by one significant figure followed by *one* zero, the square is expressed by one or two significant figures followed by *two* zeros.

In general, let d represent the digit-value of any significant figure. If that figure is followed by k zeros, the number so expressed is represented by

$$d \times 10^k.$$

The square of this number is represented by

$$(d \times 10^k)^2 = d^2 \times 10^{2k},$$

that is,

The square of a number expressed by one significant figure followed by k *zeros is a number expressed by one or two significant figures followed by* $2k$ *zeros.*

Thus, $(3000)^2 = 9000000$; $(800)^2 = 640000$.

261. This principle may be applied to find the value and place of the first figure of the square root of a given number.

ILLUSTRATIVE EXAMPLES. I. What is the first figure of the square root of 1369, and where is it written?

We seek in the given number the greatest number that can be represented in the form $d^2 \times 10^{2k}$. Since this number cannot occupy more places than the given number, which occupies four, the value of k in this example is 1, and the number sought is represented by $d^2 \times 10^2$. Pointing off *two* figures of the given number, we see that d^2 is to be found in 13. The greatest square in 13 is 9. Hence $d^2 = 9$, $d^2 \times 10^2 = 900$, and $d \times 10 = 30$.

13'69 (30
9 00

Ans. 3, written in the tens' place.

II. What is the first figure of the square root of 927369, and where is it written?

Here $k = 2$, and the number sought is represented by $d^2 \times 10^4$. Pointing off *four* figures, we see that d^2 represents the greatest square in 92, which is 81. Hence $d^2 \times 10^4 = 810000$, and $d \times 10^2 = 900$.

92'73'69 (900
81 00 00

Ans. 9, written in the hundreds' place.

III. What is the first figure of the square root of 1046529, and where is it written?

Here $k = 3$, and the number sought is represented by $d^2 \times 10^6$. Pointing off *six* figures, we see that d^2 represents the greatest square in 1, which is 1. Hence $d^2 \times 10^6 = 1000000$, and $d \times 10^3 = 1000$.

1'04'65'29 (1000
1 00 00 00

Ans. 1, written in the thousands' place.

262. From these examples may be derived a practical rule for finding the value and place of the first figure of the square root of a number.

Beginning at the units' place, point off the figures of the given number into periods of two figures each. The left-hand period will contain either one or two figures. Find *the greatest square in the number expressed by the figures of the left-hand period. Its square root is expressed by one figure. Write this figure, and annex to it as many zeros as there are periods following the left-hand period.*

Such is the method by which the first approach is made towards finding the square root of a number. The result so obtained is called the ***first approximation.***

263. When the square root of a number is expressed by two figures, it consists of tens and units. The tens are found in the first approximation; the units remain to be found. Let a represent the tens and b the units of the root. Then the whole root is represented by

$$a+b,$$

and the given number by

$$(a+b)^2 = a^2 + 2ab + b^2.$$

When the square root is expressed by three figures, it consists of hundreds, tens, and units. The hundreds are found in the first approximation; the tens and units remain to be found. Let a represent the hundreds, b the tens, and c the units. Then the whole root is represented by

$$a+b+c,$$

and the given number by

$$(a+b+c)^2 = a^2 + 2ab + b^2 + 2(a+b)c + c^2.$$

In general, a square root expressed by any number of figures may be represented by

$$a+b+\ldots+k+l,$$

and the given number by the square of this, or

$$a^2 + 2ab + b^2 + \ldots + 2(a+b+\ldots+k)l + l^2.$$

The value of a is found in the first approximation, and it remains to find the values of the other letters.

264. We shall now apply these formulas to complete the square roots already partially found in Article 261.

Illustrative Examples. I. To find the square root of 1369.

Since a is tens and b is units, the term a^2 is hundreds, $2ab$ is tens, and b^2 is units. Having subtracted the 9 hundreds (the value of a^2) from the given number, we unite the remaining 4 hundreds with the 6 tens of the given number. This gives 46 tens. The part of the given number which is represented by $2ab$ must be found in the 46 tens. Hence if we divide 46 tens by 6 tens (the value of $2a$), the quotient will be the value of b. The value of b is thus found to be 7 units. The value of the term $2ab$ is 42 tens, which, subtracted from the 46 tens, leaves 4 tens. We unite these 4 tens with the 9 units of the given number, and have 49 units as that part of the given number in which the value of b^2 must be found. The value of b^2 being just 49, we see that the value of b, found above, is correct.

$$\begin{array}{lcl}
a^2+2ab+b^2 &=& 13'69\,(\,37\\
a^2 &=& \underline{9}\\
&& 6\,)\,46\\
2ab &=& \underline{42}\\
&& 49\\
b^2 &=& 49
\end{array}$$

Note. Sometimes the value of b, found by dividing the tens by the value of $2a$, is too great. In such cases make it one or two less, as in the example given in the margin.

$$\begin{array}{r}
7'84\,(\,28\\
\underline{4}\\
4\,)\,38\\
\underline{32}\\
64\\
64
\end{array}$$

II. To find the tens and units of the square root of 927369.

$$\begin{array}{lcll}
a^2+2ab+b^2+2(a+b)c+c^2 &=& 92'73'69\,(\,963 & a=900\\
a^2 &=& \underline{81} & b=60\\
&& 18\,)\,117 & c=3\\
2ab &=& \underline{108} &\\
&& 93 &\\
b^2 &=& \underline{36} &\\
&& 192\,)\,576 &\\
2(a+b)c &=& \underline{576} &\\
&& 9 &\\
c^2 &=& 9 &
\end{array}$$

In this example a is hundreds, b is tens, and $2ab$ is thousands; hence b is found by dividing 117 thousands by 18 hundreds, which

gives 6 tens as the value of b. Again, the term $2(a+b)c$ is tens; hence c is found by dividing 576 tens by 192 tens, which gives 3 units as the value of c.

III. To find the hundreds, tens, and units of the square root of 1046529.

$$\begin{array}{lrl}
a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2 & =1'04'65'29\ (\ 1023 & \\
a^2 & =\underline{1} & \\
 & 2\ )\ 0 & a=1000 \\
2ab & =\underline{0} & b=\ \ 000 \\
 & 4 & c=\ \ \ \ 20 \\
b^2 & =\underline{0} & d=\ \ \ \ \ \ 3 \\
 & 20\ )\ 46 & \\
2(a+b)c & =\underline{40} & \\
 & 65 & \\
c^2 & =\underline{4} & \\
 & 204\ )\ 612 & \\
2(a+b+c)d & \underline{612} & \\
 & 9 & \\
d^2= & \underline{9} & \\
 & 0 &
\end{array}$$

265. From these examples we see, that, after the first approximation, the work of finding the square root of a given number is completed by the following rule:

Subtract the square of the first approximation from the part of the given number expressed by the figures of the first period.

To the figures of the remainder annex the first figure of the next period, and take the number so expressed for a dividend. For a divisor take twice the part of the root already found. The quotient (or sometimes the quotient diminished) is the value of the second figure of the root.

Multiply the divisor by the value of the figure just found, subtract the result from the dividend, and to the *figures of the remainder annex the other figure of the second period. From the number so expressed subtract the square of the value of the figure last found.*

To the figures of the remainder annex the first figure of the third period. Take the number so expressed for a dividend, twice the part of the root now found for a divisor, and proceed in the way already described to find the third figure of the root.

In a similar way find the fourth and following figures of the root, until all the figures of the given number have been used.

NOTE. If the divisor is not contained in the dividend, write 0 as the next figure of the root, annex 0 to the divisor, and annex the next two figures of the given number to the dividend, and then divide. The work in Example III. would be shortened in this way.

If there is no remainder, the given number is a perfect square; if there is, the root found is the approximate square root of the given number, correct to within less than 1.

266. Exercises.

Find the square root

31. Of 13373649.
32. Of 63792169.
33. Of 44361864.
34. Of 85184000.
35. Of 39816100.
36. Of 217081801.
37. Of 64096036.
38. Of 1732323601.

267. From the principles of multiplication, we know that the square of a fraction is found by squaring the numerator for the numerator of the result, and squaring the denominator for the denominator of the result. Conversely, therefore, the square root of a fraction is found by finding the square root of the numerator for the numerator of the result, and the square root of the denominator for the denominator of the result.

Thus, the square root of $\frac{16}{25}$ is $\frac{4}{5}$; of $\frac{49}{100}$, $\frac{7}{10}$.

268. If both numerator and denominator of the given fraction are perfect squares, the fraction is a perfect square, and its square root can be found exactly. If, when the fraction has been reduced to its lowest terms, either term is not a perfect square, the fraction is not a perfect square, and its square root can be found only approximately.

269. *The approximate square root of a fraction may be found to within less than the fractional unit.*

Proof. Let $\frac{a}{b}$ be the given fraction. We are to show how its square root can be found to within less than $\frac{1}{b}$.

Multiply both terms of the given fraction by b. This does not alter its value, but changes it to the form $\frac{ab}{b^2}$.

Let the greatest square in ab be represented by c^2. Then ab is greater than c^2 and less than $(c+1)^2$; and the fraction $\frac{ab}{b^2}$ has an intermediate value lying between

$$\frac{c^2}{b^2} \text{ and } \frac{(c+1)^2}{b^2}.$$

Hence the square root of $\frac{ab}{b^2}$, or of its equal $\frac{a}{b}$, has an intermediate value lying between

$$\frac{c}{b} \text{ and } \frac{c+1}{b},$$

and differs from neither of these limits by so much as they differ from each other. But they differ from each other by $\frac{1}{b}$. Hence either $\frac{c}{b}$ or $\frac{c+1}{b}$ is the approximate square root of $\frac{a}{b}$ to within less than $\frac{1}{b}$.

Thus, $\sqrt{\frac{5}{12}} = \sqrt{\frac{60}{144}} = \frac{\sqrt{60}}{12} = \frac{7}{12}$ within less than $\frac{1}{12}$, or $\frac{8}{12}$ within less than $\frac{1}{12}$.

Of these values $\frac{8}{12}$ should be chosen rather than $\frac{7}{12}$ as an answer, because $\sqrt{60}$ is nearer 8 than 7.

270. **Exercises.**

Find to within less than the fractional unit in each case the approximate square root

39. Of $\frac{5}{8}$.
40. Of $\frac{3}{10}$.
41. Of $\frac{9}{20}$.
42. Of $\frac{9}{10}$.
43. Of $\frac{49}{1000}$.
44. Of $\frac{362}{1000}$.
45. Of 0.7.
46. Of 0.429.
47. Of 0.025.

271. The approximate square root of an integer may be found to within less than any decimal unit, as 0.1, or 0.01, or 0.001 . . . by changing the integer to the fractional form, with $(10)^2$, or $(100)^2$, or $(1000)^2$ . . . for a denominator, and finding the square root of the fraction so formed.

272. ILLUSTRATIVE EXAMPLE. Find the square root of 5 to within less than 0.001.

$$\sqrt{5} = \sqrt{\frac{5 \times (1000)^2}{(1000)^2}} = \sqrt{\frac{5000000}{(1000)^2}} = \frac{\sqrt{5000000}}{1000} = 2.236.$$

Having found the square root of 5000000 to within less than 1, we divide the result, 2236, by 1000, and get 2.236, which is the approximate square root of 5 to within less than 0.001. Practically, it is only necessary to place a decimal point after the 5, annex six zeros (or suppose them to be annexed), and find the square root by the same method as was used in the case of integers. See work in the margin.

```
      5.00'00'00 ( 2.236
      4
  4 ) 10
       8
      20
       4
 44 ) 160
      132
       280
         9
446 ) 2710
      2676
        340
         36
        304
```

The number of zeros annexed should be twice the number of decimal places required to express the root with the required degree of accuracy.

273. **Exercises.**

Find to within less than 0.001 the square root

48. Of 2.
49. Of 3.
50. Of 7.
51. Of 15.
52. Of 10.
53. Of 26.
54. Of 40.
55. Of 90.
56. Of 150.

274. The approximate square root of a decimal fraction, or of a number partly integral and partly decimal, may be found on the same principle. It is only necessary to *annex zeros until the given number occupies twice as many decimal places as are required to express the root with the required degree of accuracy.*

Common fractions may be changed to decimals, and their approximate roots found to within less than any chosen decimal unit.

275. Exercises.

Find to within less than 0.001 the square root

57. Of 1.6.
58. Of 2.19.
59. Of 4.64.
60. Of 1.01.
61. Of 5.819.
62. Of 9.999.
63. Of $3\frac{7}{8}$.
64. Of $2\frac{3}{4}$.
65. Of $5\frac{2}{7}$.

Find to within less than 0.0001 the square root

66. Of 0.003.
67. Of 0.016.
68. Of 0.9.
69. Of 0.1.
70. Of 0.00001.
71. Of 0.036.

276. *When* k + 1 *figures of the square root of a number have been found by the ordinary method,* k *more figures may be found by simple division,* 2k + 1 *being the whole number of figures in the root.*

PROOF. Represent the part of the root expressed by the $k+1$ figures already found by a, the rest of the root expressed by the k figures yet to be found by b, and the number whose square root is required by N. Then,

$$\sqrt{N} = a + b$$

$$N = a^2 + 2ab + b^2$$

$$N - a^2 = 2ab + b^2$$

$$\frac{N - a^2}{2a} = b + \frac{b^2}{2a}.$$

We shall now show that the value of the fraction $\frac{b^2}{2a}$ is so small in comparison with b, that it may be neglected.

Since b, by supposition, represents a number expressed by k figures, b is less than 10^k, and b^2 is less than 10^{2k}.

The number represented by a is, by supposition, expressed by $k+1$ significant figures followed by k zeros, or by $2k+1$ figures in all. The least number expressed by $2k+1$ figures is 10^{2k}. Hence a is not less than 10^{2k}. But b^2 is less than 10^{2k}; hence b^2 is less than a. Therefore the value of the fraction $\frac{b^2}{a}$ is less than 1, the value of $\frac{b^2}{2a}$ is less than $\frac{1}{2}$, and the error committed by neglecting this fraction is less than $\frac{1}{2}$ a unit of the lowest order in the value of b.

In the ordinary process of finding the square root, each dividend is found by annexing to the figures of the last remainder the next figure of the given number. If, instead of annexing this one figure, we annex all the rest of the figures, we shall have expressed the complete value of $N-a^2$. The divisor is always twice the part of the root already found, or $2a$.

Therefore, if any dividend is completed by annexing all the rest of the figures of the given number, we may find by simple division as many more figures of the root as we already have, less one.

277. Illustrative Example. Find the square root of 10787053453225.

In this example, the first four figures are found by the ordinary method, and the last three by simple division.

The value of the neglected term $\frac{b^2}{2a}$ in this case is $\frac{133225}{6568000} = 0.02\ldots$

Note. In finding approximate square roots, we may find three or four figures by the ordinary method, and then two or three more by simple division, thus finding five or seven figures in all, a result accurate enough for most purposes.

```
      10′78′70′53′45′32′25 ( 3284365
       9
6 )   17
      12
       58
        4
64 )  547
      512
        350
         64
656 )  2865
       2624
         2413
           16
6568 )   2397453225
         19704
          42705
          39408
           32973
           32840
             133
```

278. Exercises.

Find as far as millionths by the method last explained the approximate square root

72. Of 2.
73. Of 3.
74. Of 5.
75. Of 8.
76. Of 11.
77. Of 14.
78. Of 17.
79. Of 30.
80. Of 50.
81. Of 75.
82. Of 80.
83. Of 160.

The n^{th} Root of Numbers.

279. The n^{th} root of a number may be found by applying the principles already explained for square root; but it is necessary to re-state some of them in a more general form.

280. Since the n^{th} power of 10 is expressed by 1 followed by n zeros, or $n+1$ figures in all, and since this is the least number that is expressed by $n+1$ figures, *the* nth *power of any integer less than* 10 *is expressed by not more than* n *figures.*

This statement is illustrated by the following table, which gives the first seven powers of each of the first ten integers.

Table of Powers.

Number.	2d Power.	3d Power.	4th Power.	5th Power.	6th Power.	7th Power.
1	1	1	1	1	1	1
2	4	8	16	32	64	128
3	9	27	81	243	729	2187
4	16	64	256	1024	4096	16384
5	25	125	625	3125	15625	78125
6	36	216	1296	7776	46656	279936
7	49	343	2401	16807	117649	823543
8	64	512	4096	32768	262144	2097152
9	81	729	6561	59049	531441	4782969
10	100	1000	10000	100000	1000000	10000000

281. A number expressed by one significant figure followed by zeros may be represented by

$$d \times 10^k,$$

where d represents the digit-value of the figure, and k the number of zeros annexed to it.

The n^{th} power of such a number is represented by

$$(d \times 10^k)^n = d^n \times 10^{nk},$$

that is,

The n$^{\text{th}}$ *power of a number expressed by one significant figure followed by* k *zeros is a number expressed by from* 1 *to* n *figures followed by* n k *zeros.*

Thus, $(3000)^4 = 81000000000000$, where $n = 4$, and $k = 3$.

282. Illustrative Examples. I. Find the value and place of the first figure of the 5th root of 657748550151.

Here $n = 5$, and we seek in the given number the greatest number that can be represented in the form

$$d^5 \times 10^{5k}.$$

Since this number cannot occupy more places than the given number, which occupies twelve, the value of k cannot exceed 2, and the number is represented by

$$d^5 \times 10^{10}.$$

65'77485'50151 (200
32 00000 00000

Pointing off ten figures of the given number, we see that d^5 is to be found in 65. The greatest fifth power in 65 is $32 = 2^5$. Hence

$$d^5 \times 10^{10} = 320000000000,$$

and $$d \times 10^2 = 200.$$

Therefore the first figure required is 2, written in the hundreds' place.

II. Find the value and place of the first figure of the cube root of 361400569867.

Here $n=3$, and, since the number represented by $d^3 \times 10^{3k}$ cannot occupy more than twelve places, k cannot exceed 3, and $d^3 \times 10^9$ represents the number sought.

Pointing off nine figures of the given number, we see that d^3 represents the greatest cube in 361, which is $343 = 7^3$.

361′400′569′867 (7000
343 000 000 000

Hence $\qquad d^3 \times 10^9 = 343000000000,$

and $\qquad d \times 10^3 = 7000.$

Therefore the first figure is 7, written in the thousands' place.

283. From these examples may be derived a practical rule for making the first approximation to the n^{th} root of a number:

Beginning at the units' place, point off the figures of the given number into periods of n *figures each. The left-hand period will contain not less than* 1 *nor more than* n *figures. Find the greatest* nth *power in the number expressed by the figures of the left-hand period. The* nth *root of this power is expressed by one figure. Write this figure, and annex to it as many zeros as there are periods following the left-hand period. The number so expressed is the first approximation sought.*

284. Exercises.

Find the first approximation

84. To $\sqrt[3]{159661140625}$. 86. To $\sqrt[4]{754507653376}$.

85. To $\sqrt[3]{15700120064}$. 87. To $\sqrt[5]{13542593318343}$.

285. To finish the work of finding the n^{th} root of a number, let a represent the part of the root already found, b the part yet to be found, and N the given number. Then,

$$\sqrt[n]{N} = a + b$$

$$N = (a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \times 2}a^{n-2}b^2 + \dots$$

$$\frac{N - a^n}{na^{n-1}} = b + \frac{(n-1)b^2}{2a} + \dots$$

From this last equation, we see that an approximation to the value of b may be made by dividing $N-a^n$ by na^{n-1}. For the terms following b in the second member, being of small value in comparison with b, may be neglected. If, at the same time, we neglect all the figures after the first in the quotient found by dividing $N-a^n$ by na^{n-1}, we may be tolerably sure that we shall find the first figure of the value of b correctly. Moreover, the degree of uncertainty, which is greatest in finding the *second* figure of the root, rapidly diminishes as we go on.

286. ILLUSTRATIVE EXAMPLES. I. Find the fifth root of 657748550151.

Here $n=5$. Having found the first approximation, 200, we subtract its fifth power, 320000000000, from the given number, but in writing the remainder stop with *the first figure of the second period*. This gives 337 (omitting zeros) as that part of the given number in which the term $na^{n-1}b$ is found. For, if the value of b is expressed by a single figure, and this figure occupies the place next below the lowest figure of a, then the figures expressing the value of the term $na^{n-1}b$ occupy one place below the lowest figure of a^n. But the lowest figure of a^n always occupies the last place in a period; hence the lowest figure of $na^{n-1}b$ occupies the first place of the following period.

$$\begin{array}{rrl} & & 65'77485'50151\,(231 \\ a^5= & & \underline{32} \\ 5a^4= & 80\,) & \underline{33\,7} \\ a^5= & & \underline{64\,36343} \\ 5a^4=1399205\,) & & \underline{1\,41142\,5} \\ & & \underline{65\,77485\,50151} \\ & & 0 \end{array}$$

The value of a being 2 (omitting zeros), that of na^{n-1} (or 5×2^4) is 80. Dividing 337 by 80, we find 4 for the next figure of the root, and 240 for the second approximation. But the fifth power of 240 is greater than the given number; hence, 240 must be rejected, and 230 must be taken for the second approximation.

To make the third approximation we apply the formula anew, taking 230 as the value of a, and the part of the root now remaining to be found as the value of b. Subtracting the fifth power of

230, which is 643634300000, from the given number, and writing the remainder so far as to include *the first figure of the third period*, we find 1411425 (omitting zeros) as the part of the given number corresponding to the term $na^{n-1}b$. The value of a now being 23 (omitting zero), that of na^{n-1} (or 5×23^4) is 1399205. Dividing 1411425 by 1399205, we find 1 as the next figure of the root, and 231 for the third approximation.

The fifth power of 231 is found to be equal to the given number; hence the given number is an exact fifth power, and 231 is its fifth root.

II. Find the cube root of 361400569867.

Here $n = 3$. In the first application of the formula $a = 7000$, and $b = 100$; in the next $a = 7100$, and $b = 20$; in the last $a = 7120$, and $b = 3$.

$$\begin{array}{rrl}
 & & 361'400'569'867\ (7123 \\
a^3 = & & \underline{343} \\
3a^2 = & 147) & \underline{18\,4} \\
a^3 = & & \underline{357\,911} \\
3a^2 = & 15123) & \underline{3\,489\,5} \\
a^3 = & & \underline{360\,944\,128} \\
3a^2 = & 1520832) & \underline{456\,441\,8} \\
 & & 361\,400\,569\,867
\end{array}$$

III. Find the square root of 1046529.

Here $n = 2$, and na^{n-1} is $2a$. This example shows how the formula for the n^{th} root may be used to find the square root of a number.

$$\begin{array}{rrl}
 & & 1'04'65'29\ (1023 \\
a^2 = & & \underline{1\,00} \\
2a = & 20) & \underline{4\,6} \\
a^2 = & & \underline{1\,04\,04} \\
2a = & 204) & \underline{61\,2} \\
 & & \underline{1\,04\,65\,29} \\
 & & 0
\end{array}$$

287. From these examples we see that after the first approximation the work of finding the n^{th} root of a number may be completed by the following rule:

Subtract the n^{th} *power of the part of the root already found from the corresponding*[1] *part of the given number.*

---

[1] The first figure of the root *corresponds* to the first period of the given number, the first two figures to the first two periods, and so on.

To the figures of the remainder annex the first figure of the next period. Take the number so expressed for a dividend. For a divisor take n *times the* n — 1st *power of the part of the root already found. The quotient (or sometimes the quotient diminished) will be the value of the next figure of the root. With the part of the root now found, repeat the process already described to find another figure of the root; and so proceed until all the figures are found.*

If there is no remainder, the given number is a perfect n^{th} power; if there is, the root found is the approximate n^{th} root of the given number, correct to within less than 1.

288. Exercises.

88. Find the fifth root of 13542593318343.
89. Find the cube root of 159661140625.
90. Find the fourth root of 754507653376.
91. Find the cube root of 15700120064.
92. Find the fifth root of 418227202051.
93. Find the fourth root of 252688187761.
94. Find the cube root of 513537536512.

289. By changing "square" to "n^{th} power," and "square root" to "n^{th} root," in Arts. 267 to 273, the principles there stated become applicable to the approximate n^{th} roots of fractions, common and decimal, and other numbers not perfect n^{th} powers.

Thus, to find the approximate n^{th} root of a common fraction, $\frac{a}{b}$, to within less than the fractional unit, we multiply numerator and denominator by b^{n-1}, which changes the fraction to the form $\frac{ab^{n-1}}{b^n}$, but without altering its value, and then find the n^{th} root of the numerator to within less than 1.

Again, to find the n^{th} root of an integer or a decimal to within less than 0.1, 0.01, 0.001 . . ., we annex zeros until n times as many decimal places are occupied by the given number as are required to express the root with the required degree of accuracy.

Common fractions may be changed to decimals, and their n^{th} roots found to within less than any chosen decimal unit.

By reasoning similar to that given in Art. 275, it may be shown, in the case of the cube root, that when $k+2$ figures have been found by the ordinary method, k more figures may be found by simple division.

290. Exercises.

Find, to within less than the fractional unit in each case,

95. The cube root of $\frac{5}{8}$.
96. The cube root of $\frac{95}{100}$.
97. The fifth root of $\frac{7}{16}$.
98. The cube root of 0.08.
99. The fourth root of 0.001.

Find, to within less than 0.00001 (finding the last two figures by simple division) the cube root

100. Of 2.
101. Of 5.
102. Of 9.
103. Of $\frac{2}{3}$.
104. Of $\frac{1}{5}$.
105. Of $\frac{22}{7}$.
106. Of 128.
107. Of 200.
108. Of 900.

Find, to within less than 0.001,

109. The fifth root of 2.
110. The fourth root of $\frac{1}{17}$.
111. The cube root of $2\frac{1}{2}$.
112. The fifth root of 5.
113. The fourth root of 17.2.
114. The fifth root of 0.875.

The n^{th} Root of Monomials.

291. The n^{th} root of a monomial may be found by reversing the operation described in Art. 245. The method, therefore, is *to find the* nth *root of the numerical co-efficient, and divide the exponent of each literal factor by* n.

This may be expressed by the formula

$$\sqrt[n]{A x^p y^q \ldots} = \sqrt[n]{A}\, x^{\frac{p}{n}} y^{\frac{q}{n}} \ldots$$

where A stands for the numerical co-efficient, and x^p, y^q . . . for the literal factors.

See Arts. 118 and 119.

292. From this method of finding the n^{th} root of a monomial, it follows that ***fractional exponents*** will result whenever the exponent of any literal factor is not divisible by n.

Thus, the square root of a^4 is $a^{\frac{4}{2}}$, or a^2; but the square root of a^3 is expressed by $a^{\frac{3}{2}}$. The square root of a is expressed by $a^{\frac{1}{2}}$, the cube root by $a^{\frac{1}{3}}$, and the n^{th} root by $a^{\frac{1}{n}}$.

The fourth root of a^3 is expressed by $a^{\frac{3}{4}}$, the cube root of a^2 by $a^{\frac{2}{3}}$, and, in general, the n^{th} root of the m^{th} power of a by $a^{\frac{m}{n}}$.

Thus we see that *a fractional exponent represents a radical sign*, and that

$$\sqrt[n]{a} \text{ and } a^{\frac{1}{n}}, \text{ also } \sqrt[n]{a^m} \text{ and } a^{\frac{m}{n}},$$

are equivalent forms, meaning the same thing.

293. From the rule for signs in multiplication, it follows that the product of an odd number of positive factors is positive, and the product of an odd number of negative factors is negative. Hence, conversely,

The odd root of a positive quantity is positive, and the odd root of a negative quantity is negative.

Thus, $\sqrt[3]{+27} = +3$, $\sqrt[3]{-27} = -3$,

$\sqrt[5]{+32} = +2$, $\sqrt[5]{-32} = -2$.

294. Again, the product of an even number of positive factors is positive, and the product of an even number of negative factors is also positive. Hence, conversely,

The even root of a positive quantity may be either a positive or a negative quantity, a fact which is expressed by writing it with the double sign ±.

Thus, $\sqrt{25} = \pm 5$, that is, $+5$ or -5;

$\sqrt[4]{81} = \pm 3$, that is, $+3$ or -3;

$\sqrt{a^2 + 2ab + b^2} = \pm(a + b)$, that is, $a + b$ or $-a - b$.

295. The even root of a negative quantity can be neither a positive nor a negative quantity; it is called an ***imaginary quantity.***

Thus, $\sqrt{-25}$, $\sqrt[4]{-81}$, $\sqrt[6]{-1}$, are symbols of imaginary quantities.

In some of the higher branches of mathematics these imaginary quantities play an important part; in elementary algebra it is enough to observe that they usually indicate some absurdity or impossibility in the question from which they arise.

296. Exercises.

115. Find the square root of $81 a^4 x^2 y^6$.
116. Find the cube root of $125 m^3 y^6$.
117. Find the cube root of $-343 a^9 x^{12}$.
118. Find the fourth root of $\frac{81 a^8 x^4}{256 b^4 y^{12}}$.

Find the value of

119. $\sqrt[3]{-512 a^6 y^9}$.
120. $\sqrt[4]{10000 x^8 y^{12}}$.
121. $\sqrt[5]{\frac{32 m^{10} n^5 x^{10}}{-3125 y^{15}}}$.
122. $(-729 a^6 x^{12})^{\frac{1}{3}}$.
123. $(-27 a^6 x^9)^{\frac{2}{3}}$.
124. $\sqrt[4]{a^{4n} b^{4n-4}}$.
125. $\sqrt[m]{a^{mp} b^{mq}}$.
126. $\sqrt{\frac{(a + x)^{2n} b^{2n+2}}{a^2 x^{2n-2}}}$.
127. $(36 x^{4n+2} y^{2m})^{\frac{1}{2}}$.
128. $(256 b^{4n} y^8)^{\frac{3}{4}}$.
129. $\sqrt[3]{0.064 m^6 n^{12}}$.
130. $\sqrt{\frac{4}{25} x^4 y^6 z^2}$.
131. $\sqrt{\frac{1}{a^{2m}}}$.
132. $(-32 z^{10})^{\frac{3}{5}}$.
133. $(-1024 y^5 z^{10})^{\frac{3}{5}}$.

297. When a monomial is not a perfect power of the same degree as the root required, it may be possible to separate it into two factors, one of which is a perfect power of that degree. The root of this factor may then be taken, and the root of the other indicated by the radical sign.

Thus,
$$\sqrt{98} = \sqrt{49 \times 2} = \sqrt{49} \times \sqrt{2} = 7\sqrt{2}.$$
$$\sqrt[3]{250\,a^7x^4} = \sqrt[3]{125\,a^6x^3 \times 2\,ax} = 5\,a^2x\sqrt[3]{2\,ax}.$$
$$\sqrt{\tfrac{1}{3}} = \sqrt{\tfrac{3}{9}} = \sqrt{\tfrac{1}{9} \times 3} = \tfrac{1}{3}\sqrt{3}.$$
$$\sqrt{-25} = \sqrt{25 \times (-1)} = 5\sqrt{-1}.$$

This operation is called *taking factors out from under the radical sign.*

298. Exercises.

As far as possible, take factors out from under the radical sign in the following examples:

134. $\sqrt{75}$.
135. $\sqrt{8\,a^3b^5}$.
136. $\sqrt[3]{\tfrac{3}{4}}$.
137. $\sqrt{-100}$.
138. $\sqrt[3]{-128\,a^7b^5}$.
139. $\sqrt{72\,x^{2n+1}}$.
140. $\sqrt{\tfrac{2}{3}x^5y^3z^4}$.
141. $\sqrt[4]{-1250}$.
142. $\sqrt[5]{a^7b^{11}}$.
143. $\sqrt{\dfrac{2n+1}{2}}$

299. The converse operation, that of *putting factors under the radical sign,* is sometimes useful.

Thus,
$$7\sqrt{2} = \sqrt{49} \times \sqrt{2} = \sqrt{49 \times 2} = \sqrt{98}.$$
$$\tfrac{1}{2}\sqrt{5} = \sqrt{\tfrac{1}{4}} \times \sqrt{5} = \sqrt{\tfrac{5}{4}} = \sqrt{1.25}.$$
$$a\sqrt[n]{x} = \sqrt[n]{a^n} \times \sqrt[n]{x} = \sqrt[n]{a^nx}.$$

300. Exercises.

Put under the radical sign the factors written before it in the following examples:

144. $3\sqrt{5}$.
145. $2\sqrt[3]{2}$.
146. $\tfrac{1}{2}\sqrt{\tfrac{4}{3}}$.
147. $2x^n\sqrt{3x}$.
148. $3a\sqrt[5]{a^2}$.
149. $\tfrac{1}{2}\sqrt{4n+2}$.

The n^{th} Root of Polynomials.

301. From the statement made in Art. 110, under Ex. II., it follows that

If the terms of a polynomial root and the terms of its n^th^ *power are arranged according to the descending (or ascending) powers of some literal factor common to them all, the first term of the power is the* n^th^ *power of the first term of the root.*

Hence, conversely,

If the terms of a given polynomial are arranged in this way, the n^th^ *root of its first term is the first term of its* n^th^ *root.*

302. Now let P stand as an abbreviation for a polynomial whose n^{th} root is to be found, A for the part of the root already found, and B for the rest of it. Then

$$P=(A+B)^n=A^n+nA^{n-1}B+\ldots$$
$$P-A^n=nA^{n-1}B+\ldots$$
$$\frac{P-A^n}{nA^{n-1}}=B+\ldots$$

Hence, to find the rest of the root, *raise the part already found to the* n^{th} *power, subtract the result from the given polynomial, and divide the remainder by* n *times the* $\text{n}-1^{\text{st}}$ *power of the part of the root already found.*

303. In applying this rule, we suppose the terms of P, of A, and of B to be arranged according to the powers of some literal factor. When the terms are so arranged, it is only necessary to divide the first term of the remainder $P-A^n$ by the first term of nA^{n-1} to get the first term of B. The *first* term of B is all that is needed; for, when this term is united to the part of the root already found, a new value results for the known part of the root, with which the steps just taken may be repeated. This process goes on until there is either no remainder, or a remainder having no term that contains the first term of nA^{n-1}.

304. ILLUSTRATIVE EXAMPLE. Find the fourth root of

$$16x^8-64x^7+192x^6-352x^5+520x^4-528x^3+432x^2-192x+81.$$

The work may be arranged as follows:

$$\begin{array}{l|l}
16x^8-64x^7+192x^6-352x^5+520x^4-528x^3+432x^2-192x+81 & \underline{2x^2-2x+3} \\
\underline{16x^8 \qquad\qquad\qquad} & 32x^6 \\
\quad\underline{-64x^7+\ \ldots \qquad\qquad\qquad\qquad\quad} & \\
\underline{16x^8-64x^7+\ 96x^6-\ 64x^5+\ 16x^4} & \\
\qquad\qquad\quad\underline{+\ 96x^6\ \ldots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} & \\
\underline{16x^8-64x^7+192x^6-352x^5+520x^4-528x^3+432x^2-192x+81} & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 &
\end{array}$$

We find the fourth root of $16x^8$, which is $2x^2$, and write it at the right as the first term of the root. Subtracting $16x^8$ from the given polynomial, we write only $-64x^7$, the first term of the remainder. For a divisor we take $4(2x^2)^3$, or $32x^6$. Dividing $-64x^7$ by $32x^6$ gives $-2x$ for the next term of the root. We now raise $2x^2-2x$ to the fourth power, and subtract the result from the given polynomial, writing only $+96x^6$, the first term of the remainder. For a divisor we should, by the rule, take $4(2x^2-2x)^3$, but we need to use only the first term of this, which is $32x^6$, or the divisor already used. Dividing $+96x^6$ by $32x^6$ gives $+3$ for the next term of the root. We now raise $2x^2-2x+3$ to the fourth power, and find the result equal to the given polynomial.

305. Exercises.

150. Find the fourth root of

$$81a^8-216a^7x+336a^5x^3-56a^4x^4-224a^3x^5+64ax^7+16x^8.$$

151. Find the fifth root of

$$y^{10}-10y^9+25y^8+40y^7-190y^6-92y^5+570y^4+360y^3-675y^2-810y-243.$$

152. Find the cube root of

$$x^6-6x^5+3x^4+28x^3-9x^2-54x-27.$$

153. Find the square root of

$$9a^6+12a^5+10a^4-2a^3-3a^2-2a+1.$$

154. Find the cube root of

$$x^9+6x^8+6x^7-4x^6+36x^5+24x^4-56x^3+144x^2-96x+64.$$

155. Find the square root of

$$9a^4-30a^3x+49a^2x^2-40ax^3+16x^4.$$

156. Find the cube root of

$$a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3.$$

306. Miscellaneous Exercises.

157. Write out seven terms in the development of $(a+b)^m$.

158. Write down the tenth term of $(x+y)^{16}$.

159. Write down the eighth term of $(a-b)^{15}$.

160. Write down the middle term of $(x-y)^{12}$.

161. Write down the r^{th} term in the development of $(a+b)^m$. Use the expression as a formula for working the last three exercises.

162. Go through the reasoning in Art. 249, writing out six terms instead of four in the right-hand member of the equation there used, and making the additional reductions after multiplying.

163. Is finding the square root of a quantity and then finding the cube root of that result equivalent to finding the sixth root? Why?

164. Find the second power of $1-\sqrt{5}$.

165. Find the cube of $2+\sqrt{2}$.

166. Raise $2+\sqrt{3}$ to the fourth power.

167. Find the second power of $\sqrt{a}+\sqrt{b}$; the third power.

168. Raise $1+2x+x^2$ to the third power and then find the square root of the result.

169. Raise 16 to the third power and then find the square root of the result. What, then, is the value of $16^{\frac{3}{2}}$?

170. What is the value of $8^{\frac{2}{3}}$? Of $16^{\frac{5}{4}}$? Of $5^{\frac{2}{3}}$? Of $243^{\frac{3}{5}}$? Of $729^{\frac{5}{6}}$? Of $5^{\frac{3}{2}}$? Of $1024^{\frac{7}{10}}$? Of $2187^{\frac{4}{7}}$? Of $7^{\frac{2}{3}}$? Of $5^{\frac{3}{4}}$?

SECTION IX.

BINOMIAL EQUATIONS.

307. In this section we shall treat of a class of equations which by clearing of fractions and uniting similar terms may be reduced to the form

$$A\,x^n = M.$$

Such equations are called ***binomial equations.***

The simple equations in Sect. II. are binomial equations in which the value of n is 1. Such equations are said to be of the first degree. The equation is of the second degree when $n = 2$, of the third degree when $n = 3$, and so on.

308. If we divide by A each member of the equation

$$A\,x^n = M,$$

we have

$$x^n = \frac{M}{A},$$

and, by taking the n^{th} root of each member,

$$x = \sqrt[n]{\frac{M}{A}},$$

which may be used as a formula for solving binomial equations.

309. Illustrative Examples. I. Solve the equation $3\,x^2 = 48$.

Since the given equation is satisfied by substituting either $+4$ or -4 for x, the equation is said to have two roots.

$$\begin{aligned} 3\,x^2 &= 48 \\ x^2 &= 16 \\ x &= \pm 4. \end{aligned}$$

II. Solve the equation $x^4 = 81$.

In this example we find four roots. These are $+3$, -3, $+3\sqrt{-1}$, and $-3\sqrt{-1}$. The first two are *real* roots, the last two *imaginary*.

$$\begin{aligned} x^4 &= 81 \\ x^2 &= \pm 9 \end{aligned}$$

$$x^2 = +9 \qquad x^2 = -9 = 9\,(-1)$$

$$x = \pm 3 \qquad x = \pm 3\sqrt{-1}.$$

III. Solve the equation $x^6 = 64$.

$$x^6 = 64$$
$$x^3 = \pm 8$$
$$x^3 = +8 \qquad x^3 = -8$$
$$x = +2 \qquad x = -2$$

In this example we find two roots, $+2$ and -2. There are four other roots to this equation, but the finding of them would require the use of principles not yet explained. It may be stated, however, that, in general, any equation of the n^{th} degree has n roots.

NOTE. A binomial equation may result from elimination, as in Exs. 10 and 11 of the following Article.

310. Exercises.

Solve the following equations:

1. $5x^2 = 2880$.
2. $2x^3 = 118638$.
3. $3x^3 = -902289$.
4. $5x^4 = 103680$.
5. $4x^2 + 3x = x^2 + 3x + 75$.
6. $\dfrac{x^3+x+8}{x^3+4} + \dfrac{x^3+x-8}{x^3-4} = 2$.
7. $\dfrac{4x-8}{x-2} = \dfrac{x+2}{4x+8}$.
8. $\dfrac{x^2-a}{x^2-b} = \dfrac{b}{a}$.
9. $\sqrt{2x^2+4x} = x + 2$.
10. $x^5 + y^3 = 368$.
    $x^5 - y^3 = 118$.
11. $x^2y = 112$.
    $y^2z = 588$.
    $z^2x = 576$.

311. The equation $Ax^n = M$ may be solved when n has a fractional value. See Art. 292.

ILLUSTRATIVE EXAMPLE. Solve the equation $3x^{\frac{2}{3}} = 12$.

FIRST SOLUTION.

$$3x^{\frac{2}{3}} = 12,$$
$$x^{\frac{2}{3}} = 4,$$
$$x^{\frac{1}{3}} = \pm 2,$$
$$x = (\pm 2)^3 = \pm 8.$$

SECOND SOLUTION.

$$3x^{\frac{2}{3}} = 12,$$
$$x^{\frac{2}{3}} = 4,$$
$$x^2 = 64,$$
$$x = \pm 8.$$

Here we divide each member of the given equation by 3, and get $x^{\frac{2}{3}} = 4$. We may either take the square root of each member of this equation, as in the first solution above, or cube each member, as in the second solution. The next step in the former is to cube each member, and in the latter to get the square root of each member.

The first solution is the more convenient when the second member is a perfect power of the degree denoted by the *numerator* of the exponent of the unknown quantity, as in the equation $x^{\frac{3}{4}} = 125$, where 125 is a perfect cube. In other cases the second solution should be followed.

312. Exercises.

Solve the following equations:

12. $4x^{\frac{3}{4}} = 500$.
13. $5x^{\frac{3}{4}} = 900$.
14. $3x^{\frac{3}{2}} = 81$.
15. $2x^{\frac{4}{3}} = 50$.
16. $4\sqrt[3]{x^2} = 62500$.
17. $5\sqrt[5]{x^4} = 1000$.
18. $\sqrt{3x^3} = 90$.
19. $\sqrt{x^{\frac{3}{5}}} = 2\sqrt{2}$.

313. Problems leading to Binomial Equations.

20. A bin to contain 20 bushels of grain is to be made $2\frac{1}{2}$ times as long as wide, and $\frac{7}{8}$ as wide as deep. What must be the dimensions?

NOTE. One cubic foot is equal to $\frac{4}{5}$ of a bushel.

21. A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy coming in sight the front was increased by 845 men, and by this movement the detachment was drawn up in five lines. How many men in the detachment?

22. A certain number is expressed by two figures. The digit-value of the left-hand figure is double that of the right-hand figure. If the figures should interchange places, the number then expressed when multiplied by the original number would give 2268 for a product. What is the number?

23. What is the number whose half and third multiplied together give 864 as a product?

24. By what number must 241200 be divided, that the quotient may be equal to the divisor, and the remainder be 119?

25. What number is that whose quarter multiplied by its square gives 2000?

26. What two numbers are they whose product is 750, and whose quotient, when the greater is divided by the less, is $3\frac{1}{3}$?

27. What two numbers are they whose product is p and whose quotient is q?

28. The product of two numbers is 1728, and their ratio that of 3 to 4. What are the numbers?

29. The product of two numbers is p, and their ratio that of a to b. What are the numbers?

30. What two numbers are they of whose squares the sum is 9818, and the difference 840?

31. What two numbers are they of whose squares the sum is a, and the difference b?

32. What two numbers are they whose ratio is that of 3 to 5, and the sum of whose squares is 14994?

33. What two numbers are they whose ratio is that of m to n, and the sum of whose squares is a?

34. The last question, changing *sum* to *difference*.

35. Three numbers are in the ratio of 3, 5, and 8, and their product is 41160. What are the numbers?

36. Three numbers are in the ratio of a, b, and c, and their product is p. What are the numbers?

37. Find three numbers such that the product of the first and second is 48, that of the second and third 112, and that of the third and first 84.

38. The area of a given circle is 270 square feet. Find the diameter.

NOTE. The area of a circle is expressed by πr^2, where r is the length of the radius. The Greek letter π (pronounced *pi*) is used by mathematical writers to represent the ratio of the circumference to the diameter of a circle. The circumference and diameter being incommensurable, the numerical value of this ratio can only be expressed approximately. The value $\frac{22}{7}$ is near enough for many purposes. The value 3.1416 is correct to within less than 0.00001, and will be used in this book.

39. Find how many feet long the rope must be by which a cow is tethered that she may graze over just half an acre of ground.

40. A cistern is to be built in the form of a cylinder capable of holding 9000 gallons; and its depth is to be equal to its diameter. Find the dimensions.

NOTE. A gallon contains 231 cubic inches.

41. The height of a given cone is to the diameter of the base as 2 to 3. If the volume of this cone is 360 cubic feet, what is its height?

NOTE. The volume of a cone is expressed by $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h the height of the cone.

42. What is the diameter of a sphere whose volume is 1000 cubic feet?

NOTE. The volume of a sphere is expressed by $\frac{4}{3}\pi r^3$, where r is the radius.

43. Find the surface of a sphere whose volume is 1000 cubic feet.

NOTE. The surface of a sphere is expressed by $4\pi r^2$ where r is the radius.

44. Find the diameter of a leaden ball weighing a ton (2000 pounds).

NOTE. A cubic foot of lead weighs 712 pounds.

45. Find the diameter of a sphere whose surface is 100 square feet.

46. Find the volume of a sphere whose surface is 25 square feet.

47. A cubic foot of water weighs 1000 ounces. The specific gravity of ice is 0.918. Find the diameter of a sphere of ice weighing a ton.

SECTION X.

QUADRATIC EQUATIONS.

314. If, when cleared of fractions and reduced to its simplest form, an equation contains the second but no higher power of the unknown quantity, it is called a ***quadratic equation,*** or an ***equation of the second degree.***

315. The general form to which an equation of the second degree may be reduced is

$$Ax^2 + Bx = M.$$

When the term containing the first power of the unknown quantity is wanting, the equation is of the form

$$Ax^2 = M,$$

and is called a ***pure quadratic.***

316. When the term containing the first power and the term containing the second power of the unknown quantity are both present, the equation is called an ***affected quadratic.***

317. A pure quadratic, being a binomial equation, is solved by the method explained in the last section. The following examples will illustrate the method of solving affected quadratics.

318. Illustrative Examples. I. Solve the equation $x^2 + 6x = 27$.

The first step in this solution is to make the first member of the equation a perfect square. We know (Art. 128) that the quantity $x^2 + 2ax$ is made a perfect square by adding a^2 to it; for the result $x^2 + 2ax + a^2$ is the square of $x + a$. Now the first member of the given equation is of the form $x^2 + 2ax$, where $2a$ represents 6.

$$\begin{aligned} x^2 + 6x &= 27 \\ x^2 + 6x + 9 &= 36 \\ x + 3 &= \pm 6 \\ x &= -3 \pm 6 \\ x_1 &= +3 \\ x_2 &= -9. \end{aligned}$$

Hence $a = 3$, $a^2 = 9$, and the first member will be made a perfect square by adding 9 to it.

When 9 has been added to each member of the given equation, the square root of each member is taken, that of the second member being written with the double sign ±. Thus we have two simple equations,

$$x + 3 = +6 \text{ and } x + 3 = -6,$$

from which to find values of x. Subtracting 3 from each member of both, we find

$$\begin{aligned} x &= -3 + 6 & \text{and} \qquad x &= -3 - 6 \\ &= +3. & &= -9. \end{aligned}$$

The given equation is satisfied when either $+3$ or -9 is substituted for x therein. For

$$3^2 + 6 \times 3 = 27, \text{ and } (-9)^2 + 6 \times (-9) = 27.$$

Every quadratic equation has two roots, one found by taking the positive value of the square root of the second member, and the other by taking the negative value.

These two roots will henceforth be represented by the letters x_1 and x_2. In the present example $x_1 = +3$ and $x_2 = -9$.

II. Solve the equation $x^2 - 20x = 300$.

Here the first member is of the form $x^2 - 2ax$, where $-2a$ represents -20. Hence $a = 10$, $a^2 = 100$, and the first member becomes a perfect square by adding 100 to each member.

$$\begin{aligned} x^2 - 20x &= 300 \\ x^2 - 20x + 100 &= 400 \\ x - 10 &= \pm 20 \\ x &= 10 \pm 20 \\ x_1 &= 30 \\ x_2 &= -10. \end{aligned}$$

III. Solve the equation $3x^2 + 7x = 6$.

Here we divide each member by 3 to reduce the first member to the form $x^2 + 2ax$, and find that $2a$ represents $\frac{7}{3}$. Hence $a = \frac{7}{6}$, and $a^2 = \frac{49}{36}$, which is added to each member.

$$\begin{aligned} 3x^2 + 7x &= 16 \\ x^2 + \tfrac{7}{3}x &= \tfrac{16}{3} \\ x^2 + \tfrac{7}{3}x + \tfrac{49}{36} &= \tfrac{16}{3} + \tfrac{49}{36} = \tfrac{241}{36} \\ x + \tfrac{7}{6} &= \tfrac{1}{6}\sqrt{241} \\ x &= -\tfrac{7}{6} \pm \tfrac{1}{6}\sqrt{241} \\ x_1 &= 1.4207 \\ x_2 &= -3.7540. \end{aligned}$$

When, as in this example, approximate roots have to be found, results will be given *to the nearest ten-thousandth.*

319. The method illustrated by the foregoing examples may be generalized by applying it to solve the general equation,

$$Ax^2 + Bx = M.$$

Dividing each member by A,

$$x^2 + \frac{B}{A}x = \frac{M}{A}.$$

Completing the square by adding $\left(\frac{B}{2A}\right)^2$, or $\frac{B^2}{4A^2}$, to each member,

$$x^2 + \frac{B}{A}x + \frac{B^2}{4A^2} = \frac{4AM + B^2}{4A^2}.$$

Taking the square root of each member,

$$x + \frac{B}{2A} = \pm\frac{\sqrt{4AM + B^2}}{2A}.$$

Finally, subtracting $\frac{B}{2A}$ from each member, we have the result,

$$x = -\frac{B}{2A} \pm \frac{\sqrt{4AM + B^2}}{2A},$$

which may be used as a general formula for the solution of quadratic equations.

320. Application. Applying this formula to solve the examples in the last article, we have:

In I, $A = 1$, $B = 6$, and $M = 27$, substituting which, we get

$$x = -\frac{6}{2} \pm \frac{\sqrt{108 + 36}}{2}$$

$$= -3 \pm 6 = +3 \text{ or } -9.$$

In II, $A = 1$, $B = -20$, and $M = 300$, substituting which, we get

$$x = -\frac{-20}{2} \pm \frac{\sqrt{1200 + (-20)^2}}{2}$$

$$= 10 \pm 20 = +30 \text{ or } -10.$$

In III, $A = 3$, $B = 7$, and $M = 16$, substituting which, we get

$$x = -\frac{7}{6} \pm \frac{\sqrt{192 + 49}}{6} = -\tfrac{7}{6} \pm \tfrac{1}{6}\sqrt{241}.$$

Exercises in solving Quadratic Equations.

321. The following equations may be solved by the method illustrated in Art. 318, or by substituting in the formula as was done in Art. 320.

1. $x^2 - 8x = -15.$
2. $x^2 - 7x + 12 = 0.$
3. $x^2 - 3x = 4.$
4. $x^2 - 10x + 16 = 0.$
5. $x^2 - x - 30 = 160.$
6. $3x^2 - 9x - 2 = 82.$
7. $7x^2 - 21x + 13 = 293.$
8. $x^2 + 2x = 6.$
9. $x^2 + 5x = 10.$
10. $x^2 - 7x = 7.$
11. $900 - 30x = x^2.$
12. $1 - x = x^2.$
13. $\frac{x^2}{3} + \frac{4x}{5} - 19 = 15\tfrac{1}{5}.$
14. $\frac{2x^2}{3} + 1\tfrac{1}{2} = \frac{x}{2} + 6.$
15. $x^2 - 0.3x = 0.7.$
16. $0.2x^2 + 0.39x = 1.245.$
17. $x + 4 + \frac{7x - 8}{x} = 13.$
18. $\frac{x + 11}{x} + \frac{9 + 4x}{x^2} = 7.$
19. $14 + 4x - \frac{x + 7}{x - 7} = 3x + \frac{9 + 4x}{3}.$
20. $\frac{x + 3}{2} + \frac{16 - 2x}{2x - 5} = 5\tfrac{1}{5}.$
21. $\frac{4x + 7}{9} - 1 = \frac{x + 4}{3} - \frac{7 - x}{x - 3}.$
22. $7x - \frac{23x + 60}{7} = \frac{15 - x}{4} - \frac{12 - 3x}{4x - 5}.$
23. $3 + \frac{3x - 16}{18} = \frac{2x + 9}{9} + \frac{4x - 3}{4x + 3}.$
24. $\frac{6}{x + 1} + \frac{2}{x} = 3.$
25. $\frac{40}{x - 5} + \frac{27}{x} = 13.$
26. $\frac{3x - 7}{x} + \frac{4x - 10}{x + 5} = 3\tfrac{1}{2}.$
27. $\frac{8x}{x + 2} - 6 = \frac{20}{3x}.$
28. $\frac{x}{x + 60} = \frac{7}{3x - 5}.$
29. $\frac{2x}{x - 4} + \frac{2x - 5}{x - 3} = 8\tfrac{1}{3}.$
30. $(x + 6)(4x + 5) = (2x + 4)(4x - 10).$
31. $2x + 3 : 2x + 9 = 8 + x : 4 + 3x.$
32. $(2x - 5)(3x + 4) - (3x + 3)(x - 4) = 153.$

322. Problems to be solved by the Use of Quadratic Equations with One Unknown Quantity.

33. The floor of a hall is 20 feet longer than it is wide, and its area is 2400 square feet. Find the dimensions.

34. One number is 6 greater than another, and their product is 91. Find the numbers.

35. Find two numbers whose difference is d, and whose product is p.

36. What number added to its square gives 870 for the sum?

37. What number added to its square root gives 210 for the sum?

HINT. Let x^2 represent the number.

38. What number exceeds its square root by 992?

39. A rectangular field containing three acres is one yard longer than it is wide. Find its dimensions.

40. Find two consecutive numbers whose product is 156.

NOTE. Consecutive numbers are those that follow one another in the natural series 1, 2, 3, 4, &c.

41. Find three consecutive numbers whose sum is equal to the product of the first two.

42. The sum of the squares of two consecutive numbers is 481. Find the numbers.

43. One number is double another, and their product added to their sum gives 90. Find the numbers.

44. A horse was sold for $119, and the gain was as many *per cent* as the horse cost dollars. What did the horse cost?

HINT. If x represents the number of dollars the horse cost, x per cent, or $\frac{x}{100}$, represents the per cent of gain, and $\frac{x^2}{100}$, the gain.

45. A rectangular garden is $\frac{2}{3}$ as wide as it is long, and the number of square rods in it exceeds the number of rods around it by 1376. What are the dimensions?

46. There is a rectangular garden whose length is to its breadth as 3 to 2. Within the fence surrounding it, there is

a paved walk $4\frac{1}{2}$ feet wide, and the area left for cultivation is 146481 square feet. What is the length and the breadth of the garden?

47. The hypothenuse of a certain right-angled triangle is 2 feet longer than the base and 9 feet longer than the perpendicular. Find the sides of the triangle.

NOTE. The square on the hypothenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

48. Divide a straight line 10 inches long into two parts such that the longer part may be a mean proportional between the whole line and the shorter part.

49. Generalize the last problem, letting a represent the length of the line to be divided.

50. Separate the number 40 into two parts such that the sum of their cubes may be 19000.

51. Separate the number a into two parts such that the sum of their cubes may be s.

52. A girl bought a number of oranges for 40 cents. Had she bought at another place, she would have received three more oranges for the same money, each orange costing $\frac{2}{3}$ of a cent less. How many did she buy?

53. A messenger having finished a journey of 108 miles says that he can return in 6 hours' less time by riding 3 miles an hour faster. How many miles an hour did he ride?

54. A dealer bought a number of horses for $1120 and sold them at an average price of $157.50 each, thereby gaining on the whole as much as one horse cost him. How many horses did he buy?

55. A and B together can do a piece of work in 4 days, but it would take A alone 6 days longer than it would take B alone. How long would it take each alone to do the work?

56. A merchant paid $816 for flour. Had the price been half a dollar less a barrel, he would have received 6 more barrels for the same money. How many barrels did he buy?

57. A ship's crew, being on short allowance, shared equally 84 ounces of meat a day; but, after 4 men had died,

the same daily allowance of meat gave each man half an ounce more. How many men were there in the crew?

58. A drover bought a number of sheep for $300. Out of these he kept 15, and, selling the rest for $270, gained half a dollar on each one sold. How many sheep did he buy?

59. What is the price of oranges when eight more in a dollar's worth would lower the price five cents a dozen?

60. Twenty persons contribute $48 to meet the expense of a picnic, one half of the whole being contributed in equal portions by the women and the other half by the men; but each man gave a dollar more than each woman. How many men were there, and how many women? How much did each person contribute?

61. Two railway trains start at the same time to run 156 miles; but one runs a mile an hour faster than the other, and arrives at its journey's end one hour sooner. At what rate does each train run?

62. Two boys together carried 100 eggs to market; and, though one had more eggs than the other, each received the same amount of money. The first said to the second, "At my selling-price, I should have got 90 cents for your eggs." The second replied, "But at my price, I should have got only 40 cents for yours." How many eggs did each carry?

63. Two farmers sold apples in the market. The second sold 3 barrels more than the first; and together they received $35. As they were driving home, the first said to the second, "At my price, I should have got 24 dollars for your apples." The other replied, "And at my price, I should have got for yours 12½ dollars." How many barrels did each sell?

64. Two travellers, A and B, set out at the same time from towns 100 miles apart, and travelled toward each other. When they met, A said that it would have taken him only 6 hours and 20 minutes to come as far as B had; and B acknowledged that it would have taken him 15 hours to come as far as A had. How far had each come?

Simultaneous Quadratic Equations.

323. ILLUSTRATIVE EXAMPLES. I. Solve the equations $x + y = 20$ and $xy = 96$.

First substitute in (1) an expression for y derived from (2). This gives (3), which, reduced and solved, gives 12 and 8 for values of x. By substituting these values in either original equation, we find, that, when $x = 12$, $y = 8$; and when $x = 8$, $y = 12$.

$$(1)\qquad x + y = 20$$
$$(2)\qquad xy = 96$$
$$y = \frac{96}{x}$$
$$(3)\qquad x + \frac{96}{x} = 20$$
$$x^2 + 96 = 20x$$
$$x^2 - 20x = -96$$
$$x = 10 \pm 2.$$

Roots. $x_1 = 12,\quad y_1 = 8,\quad x_2 = 8,\quad y_2 = 12.$

II. Solve the equations $x^2 + y^2 = 20$ and $x + y = 6$.

From (2) is derived an expression for y, and from that an expression for y^2, which, substituted in (1), gives (3). The two values of x found by solving (3) are substituted separately in (2) to find values of y.

$$(1)\qquad x^2 + y^2 = 20$$
$$(2)\qquad x + y = 6$$
$$y = 6 - x$$
$$y^2 = 36 - 12x + x^2$$
$$(3)\qquad x^2 + 36 - 12x + x^2 = 20$$
$$2x^2 - 12x = -16$$
$$x^2 - 6x = -8$$
$$x = 3 \pm 1.$$

Roots. $x_1 = 4,\quad y_1 = 2,\quad x_2 = 2,\quad y_2 = 4.$

III. Solve the equations $x^2 + y^2 = 61$ and $xy = 30$.

Multiply each member of (2) by 2 to get (3). Add (3) and (1) to get (4). Subtract (3) from (1) to get (5). Take the square root of each member of (4) and of (5) to get (6) and (7). Taking the double signs separately we get four sets of values for x and y as follows:

$$(1)\qquad x^2 + y^2 = 61$$
$$(2)\qquad xy = 30$$
$$(3)\qquad 2xy = 60$$
$$(4)\qquad x^2 + 2xy + y^2 = 121$$
$$(5)\qquad x^2 - 2xy + y^2 = 1$$
$$(6)\qquad x + y = \pm 11$$
$$(7)\qquad x - y = \pm 1.$$

$x + y = +11$	$x + y = +11$	$x + y = -11$	$x + y = -11$
$x - y = +1.$	$x - y = -1$	$x - y = +1$	$x - y = -1$
$x_1 = 6,$	$x_2 = 5,$	$x_3 = -5,$	$x_4 = -6,$
$y_1 = 5.$	$y_2 = 6.$	$y_3 = -6.$	$y_4 = -5.$

IV. Solve the equations $x^2 + xy - y^2 = 1$ and

$$x^2 - xy + 2y^2 = 8.$$

In this example each unknown term contains the square of one or the product of two unknown quantities. Such terms are said to be of the *second degree*. The following method of solution may be used whenever all the unknown terms of both equations are of the second degree.

Let r represent the ratio of y to x, so that $y = rx$. Substituting rx for y in the given equations,

$$(1) \qquad x^2 + rx^2 - r^2x^2 = 1$$

$$(2) \qquad x^2 - rx^2 + 2r^2x^2 = 8.$$

From (1),
$$x^2 = \frac{1}{1 + r - r^2}$$

and from (2),
$$x^2 = \frac{8}{1 - r + 2r^2}$$

whence, by equating the two expressions for x^2,

$$\frac{1}{1 + r - r^2} = \frac{8}{1 - r + 2r^2}.$$

Clearing of fractions,

$$1 - r + 2r^2 = 8 + 8r - 8r^2$$

reducing,
$$10r^2 - 9r = 7$$

and solving,
$$r = \tfrac{9}{20} \pm \tfrac{19}{20} = \tfrac{7}{5} \text{ or } -\tfrac{1}{2}.$$

Substitute $\frac{7}{5}$ for r in (1) or (2), say in (1),

$$x^2 + \frac{7x^2}{5} - \frac{49x^2}{25} = 1$$

$$x^2 = \tfrac{25}{11}$$

$$x = \pm 5\sqrt{\tfrac{1}{11}}.$$

Substitute these values of x in the equation $y = rx$, remembering that r is now $\frac{7}{5}$,

$$y = \pm 7\sqrt{\tfrac{1}{11}}.$$

Again, substitute $-\frac{1}{2}$ for r in (1).

$$x^2 - \tfrac{1}{2}x^2 - \tfrac{1}{4}x^2 = 1$$

$$x^2 = 4$$

$$x = \pm 2$$

Substitute these values of x in $y = rx$,
remembering that r is now $-\frac{1}{2}$,

$$y = -\tfrac{1}{2} \times (\pm 2)$$

$$y = \mp 1.$$

Roots.

$$x_1 = 5\sqrt{\tfrac{1}{11}},\ x_2 = -5\sqrt{\tfrac{1}{11}},\ x_3 = 2,\ x_4 = -2,$$

$$y_1 = 7\sqrt{\tfrac{1}{11}},\ y_2 = -7\sqrt{\tfrac{1}{11}},\ y_3 = -1,\ y_4 = 1.$$

324. **Exercises in solving Simultaneous Quadratic Equations.**

65. $x^2+y^2=74.$
$3x+5y=46.$

66. $x^2+xy+y^2=7.$
$2x+3y=8.$

67. $3x^2-2xy=15.$
$2x+3y=12.$

68. $\frac{x^2}{y^2}=\frac{85}{9}-\frac{4x}{y}.$
$x-y=2.$

69. $x-\frac{x-y}{2}=4.$
$y-\frac{x+3y}{x+2}=1.$

70. $\frac{5y}{x}+\frac{2y+3}{x+y}=0.$
$4x+3y=1.$

71. $2x^2+3y^2=30.$
$5x^2-4y^2=29.$

72. $x^2+y^2=65.$
$xy=28.$

73. $y^2-x^2=16.$
$xy=15.$

74. $\frac{x}{3}+\frac{y}{2}=1.$
$\frac{3}{x}+\frac{2}{y}=4.$

75. $x^3+y^3=72.$
$x+y=6.$

76. $x^3-y^3=19.$
$x-y=1.$

77. $x^2+xy=45.$
$xy-y^2=2.$

78. $x^2-xy=35.$
$xy+y^2=18.$

79. $x+y=7.$
$x^2y^2=100.$

80. $x^2y^2+4xy=96.$
$x+y=6.$

81. $\frac{x^3+y^3}{x+y}=43.$
$\frac{x^3-y^3}{x-y}=127.$

82. $x+y:x-y=4:1.$
$(x+y)(x-y)=64.$

83. $xy=77.$
$x^2-y^2:(x-y)^2=9:2.$

84. $xy=750.$
$(x+7)(y-2)=851.$

85. $4x^2-4xy+2y^2=45.$
$2x^2+3xy-3y^2=13\frac{1}{2}.$

325. Problems producing Simultaneous Quadratic Equations.

86. The sum of two numbers is 26, and the sum of their squares is 436. Find the numbers.

87. What two numbers are they whose difference is 17, and the sum of whose squares is 325?

88. What two numbers are they whose product is 255, and the sum of whose squares is 514?

89. What two numbers are they whose sum is to the greater as 11 to 7, the difference of their squares being 132?

90. The sum of two numbers is 16, and the sum of their reciprocals is $\frac{1}{3}$. What are the numbers?

91. The area of a rectangle is 108 square feet, and the length of its diagonal 15 feet. Find the length and breadth of the rectangle.

92. The area of a right-angled triangle is 30 square feet, and the length of the hypothenuse is 13 feet. How long is the base, and the perpendicular?

93. The perpendicular of a right-angled triangle is 15 feet long, and the hypothenuse is 3 feet longer than the base. How long is the base, and the hypothenuse?

94. A tailor has noticed that broadcloth on being wet shrinks up $\frac{1}{8}$ in its length and $\frac{1}{16}$ in its breadth. If the surface of a piece of broadcloth is $5\frac{3}{4}$ square yards less, and the distance around it $4\frac{1}{4}$ yards less, than before it was wet, what was the length and width of the cloth originally?

95. In going a mile the fore wheel of a carriage makes 128 revolutions more than the hind wheel; but, if the circumference of each wheel were one foot greater, the difference would be only 110 revolutions in going the same distance. What is the circumference of each wheel?

96. Find three numbers such that when the sum of the first and second is multiplied by the third, the product is 63; when the sum of the second and third is multiplied by the first, the product is 28; and, when the sum of the third and first is multiplied by the second, the product is 55.

Equations of the Quadratic Form.

326. An equation containing only two different powers of the unknown quantity may be solved as a quadratic, if the exponent of one power of the unknown quantity is double the exponent of the other. Such equations are of the general form $Ax^{2n} + Bx^n = M$.

327. ILLUSTRATIVE EXAMPLE. Solve the equation $x^6 - 7x^3 = 8$.

The equation is solved by completing the square as if x^3 were the unknown quantity. The two values of x^3 are found to be 8 and -1; and the two values of x, found by taking the cube roots of these, are 2 and -1.

There are four other roots to this equation, the method of finding which will be explained further on.

$$x^6 - 7x^3 = 8$$
$$x^6 - 7x^3 + \frac{49}{4} = 8 + \frac{49}{4} = \frac{81}{4}$$
$$x^3 - \frac{7}{2} = \pm\frac{9}{2}$$
$$x^3 = 8, \text{ or } -1$$
$$x_1 = \sqrt[3]{8} = 2$$
$$x_2 = \sqrt[3]{-1} = -1$$

328. Exercises.

97. $z^4 - 5z^2 + 4 = 0$.
98. $x^8 - 97x^4 + 1296 = 0$.
99. $x^6 + 27 = 28x^3$.
100. $x^4 + 28224 = (25x)^2$.
101. $y^6 - 14y^3 + 24 = 0$.
102. $z^6 - 19z^3 = 216$.
103. $17y^4 - 1 = 16y^8$.
104. $(13x^2)^2 + (12x)^2 = 5^2$.
105. $y^4 - 6y^2 = 45$.
106. $x^{2n} - 5x^n = 24$.
107. $(a + b + x)(a + b - x)(a - b + x)(a - b - x) = 0$.

Solution of Equations by factoring the First Member.

329. ILLUSTRATIVE EXAMPLES. I. Solve the equation $x^2 + 6x = 27$.

The first step is to reduce the given equation to a form in which the second member is 0, which is done by subtracting the second member itself from each member of the equation. Then the first

$$x^2 + 6x = 27$$
$$x^2 + 6x - 27 = 0$$
$$(x - 3)(x + 9) = 0$$
$$x - 3 = 0, \text{ whence } x = +3$$
$$x + 9 = 0, \text{ whence } x = -9$$

member is separated into two factors by the method of Art. 131. Now, any value of x which makes either of these factors equal to 0 satisfies the equation, and must therefore be a root of the equation. Hence we have two simple equations, $x-3=0$ and $x+9=0$, from which to find the roots. See Art. 318.

II. Solve the equation $x^3=8$.

The equation being reduced to the form $x^3-8=0$, we see (Art. 139) that one factor of the first member is $x-2$, the other factor being x^2+2x+4. By making $x-2=0$, we find one root of the equation; and by making $x^2+2x+4=0$ we find two more.

$$\begin{aligned} x^3&=8\\ x^3-8&=0\\ (x-2)(x^2+2x+4)&=0\\ x-2&=0, \text{ whence } x=2\\ x^2+2x+4&=0, \text{ whence } x=-1\pm\sqrt{-3}\\ x_1&=2\\ x_2&=-1+\sqrt{-3}\\ x_3&=-1-\sqrt{-3} \end{aligned}$$

The three roots here found are the three cube roots of 8. The first root, 2, is the arithmetical cube root, the other two are algebraic cube roots. That the latter are quantities, each of which raised to the third power is equal to 8, is easily shown by actual multiplication.

III. Solve the equation $x^6=64$.

$$\begin{aligned} x^6&=64\\ x^6-64&=0\\ (x^3-8)(x^3+8)&=0\\ (x-2)(x^2+2x+4)(x+2)(x^2-2x+4)&=0\\ x-2&=0, \text{ whence } x_1=2\\ x^2+2x+4&=0, \text{ whence } x_2=-1+\sqrt{-3}\\ x_3&=-1-\sqrt{-3}\\ x+2&=0, \text{ whence } x_4=-2\\ x^2-2x+4&=0, \text{ whence } x_5=1+\sqrt{-3}\\ x_6&=1-\sqrt{-3} \end{aligned}$$

The first member is separated into four factors. By making each factor equal to 0, we have four equations, two of the first degree and two of the second, giving in all six roots. This is a complete solution of the equation which was only partially solved in Art. 309, Ex. III.

330. Exercises.

Solve the following equations by factoring:

108. $x^2-2x=3$.

109. $x^2+11x+30=0$.

110. $x^2+x=56$.

111. $56=15y+y^2$.

112. $y^4+3y^2+2=0$.

113. $x^4+3x^2=88$.

114. $y^2+(a+c)y+ac=0$.

115. $y^2-(a-c)y-ac=0$.

116. Ex. 107, Art. 328, in the same way.

Find the three roots

117. Of $x^3=125$.

118. Of $x^3=-125$.

119. Of $x^3=216$.

120. Of $x^3=-\frac{1}{8}$.

121. Of $x^3=a^3$.

122. Of $x^3=-a^3$.

123. Find all the roots of the equations $x^3=8$ and $x^3=-1$, thus completing the solution of the equation in Art. 327.

Find the six roots

124. Of $x^6=729$.

125. Of $x^6=1$.

126. Of $z^6-19z^3=216$.

127. Of $x^6+27=28x^3$.

331. The first member of a quadratic equation when reduced to the form

$$x^2+px+q=0$$

is always separable into two factors of the form

$$(x-x_1)(x-x_2),$$

where x_1 and x_2 are the two roots of the equation.

Proof. We first solve the equation by completing the square,

$$x^2+px+\tfrac{1}{4}p^2=\tfrac{1}{4}p^2-q$$

$$x+\tfrac{1}{2}p=\pm\sqrt{\tfrac{1}{4}p^2-q}$$

obtaining the roots

$$x_1=-\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2-q}$$

$$x_2=-\tfrac{1}{2}p-\sqrt{\tfrac{1}{4}p^2-q}.$$

By adding the two roots we find

$$x_1+x_2=-p,$$

and by multiplying,

$$x_1x_2=q.$$

In the given equation we may substitute for p its equivalent $-(x_1+x_2)$ and for q its equivalent x_1x_2. Then the equation becomes

$$x^2-(x_1+x_2)x+x_1x_2=0,$$

which by factoring (Art. 130) becomes

$$(x-x_1)(x-x_2)=0,$$

thus proving the principle stated.

332. **This principle enables us to find a quadratic equation which shall have any roots we choose.**

Thus, to form an equation whose roots shall be 30 and -10, we have

$$(x-30)(x+10)=0.$$

Multiplying, $$x^2-20x-300=0,$$

which is the same equation that was solved in Art. 318, Ex. II.

Again, to form an equation whose roots shall be $-\frac{7}{6}+\frac{1}{6}\sqrt{241}$ and $-\frac{7}{6}-\frac{1}{6}\sqrt{241}$, we have

$$\left(x+\tfrac{7}{6}-\tfrac{1}{6}\sqrt{241}\right)\left(x+\tfrac{7}{6}+\tfrac{1}{6}\sqrt{241}\right)=0.$$

Multiplying, $$x^2+\tfrac{14}{6}x+\tfrac{49}{36}-\tfrac{241}{36}=0$$

$$3x^2+7x-16=0.$$

which is the same equation that was solved in Art. 318, Ex. III.

333. Exercises.

Form equations whose roots shall be

128. 5 and 3.
129. 7 and -4.
130. $\frac{3}{4}$ and $\frac{3}{5}$.
131. $3\frac{1}{2}$ and $5\frac{1}{2}$.
132. 5 and -5.
133. $2+\sqrt{3}$ and $2-\sqrt{3}$.
134. $5+3\sqrt{5}$ and $5-3\sqrt{5}$.
185. $a+\sqrt{b}$ and $a-\sqrt{b}$.
136. $k+l$ and $k-l$.
137. $\frac{1}{2}h+\frac{1}{2}\sqrt{m}$ and $\frac{1}{2}h-\frac{1}{2}\sqrt{m}$.
138. $-1+\sqrt{-3}$ and $-1-\sqrt{-3}$.
139. $-1+\sqrt{-1}$ and $-1-\sqrt{-1}$.
140. $s+\sqrt{s^2-p^2}$ and $s-\sqrt{s^2-p^2}$.

NOTE. This principle is equally applicable to equations of higher degrees. We may form an equation having as many given roots as we please, by subtracting each root from x, multiplying together the remainders, and placing the product equal to 0.

Thus an equation having three roots, 2, 3, and -3, is

$$(x-2)(x-3)(x+3)=0, \text{ or } x^3-2x^2-9x+18=0.$$

Conversely, an equation of any degree having 0 for one member can be solved by the methods explained above, if the other member is separable into factors $(x-x_1)$, $(x-x_2)$, &c.

SECTION XI.

THEORY OF EXPONENTS.

334. The first definition of an exponent, that it is a number indicating how many times a quantity is used as a factor (Art. 79), applies only to positive integral exponents. The law of exponents based on this definition, and expressed by the formula

$$a^m \times a^n = a^{m+n},$$

is originally subject to the same restriction; for it was derived (Art. 80) from examples in which m and n had only positive integral values.

We shall now remove this restriction, letting m and n have any values, negative as well as positive, and fractional as well as integral, and then inquire what meanings should be given to negative and fractional numbers used as exponents, that the law of exponents expressed by the formula may hold good universally.

335. *What should be the meaning of zero used as an exponent?*

In the formula $a^m \times a^n = a^{m+n}$, let $n = 0$.

Then $$a^m \times a^0 = a^{m+0} = a^m$$

from which it is plain that a^0 is a quantity, which, used as a multiplier, makes the product equal to the multiplicand. The only quantity of which this is true is 1.

Hence $$a^0 = 1.$$

This equation defines the meaning of zero used as an exponent. (See Art. 83.)

336. *What should be the meaning of a negative number used as an exponent?*

In the formula, let m have a positive value, say $+5$, and give n first a positive and then a negative value, say $+3$, then -3.

Then (1) $a^5 \times a^3 = a^{5+3} = a^8$
(2) $a^5 \times a^{-3} = a^{5-3} = a^2$

from which it is plain that opposite effects are wrought by using a^3 and a^{-3} as multipliers. In (1) there are three *more* equal factors in the product than in the multiplicand: in (2) there are three *fewer*. Hence (Art. 83) multiplying by a^{-3} is the same as dividing by a^3, that is,

$$a^5 \times a^{-3} = a^5 \div a^3 = a^5 \times \frac{1}{a^3},$$

and $$a^{-3} = \frac{1}{a^3}.$$

In general, the meaning of a negative exponent is defined by the equation

$$a^{-n} = \frac{1}{a^n},$$

that is, a^{-n} and a^n are reciprocals of each other.

From the above equation we infer that

$$a^n \times a^{-n} = 1,$$

and that $$\frac{1}{a^{-n}} = a^n.$$

337. *What should be the meaning of a positive fraction used as an exponent?*

In the formula, let $m = n = \frac{1}{2}$.

Then $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a,$

that is, $a^{\frac{1}{2}}$ is one of *two* equal factors which multiplied together produce a. In other words, $a^{\frac{1}{2}}$ is the *square root* of a, or $a^{\frac{1}{2}} = \sqrt{a}$.

In the same way,

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a.$$

Whence $a^{\frac{1}{3}}$ is the *cube root* of a, or $a^{\frac{1}{3}} = \sqrt[3]{a}$.

And, in general, $a^{\frac{1}{n}}$ is one of n equal factors whose product is a.

Hence the defining equation

$$a^{\frac{1}{n}} = \sqrt[n]{a},$$

as in Art. 292.

Again, since

$$a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = a^{3},$$

$a^{\frac{3}{4}}$ is one of *four* equal factors whose product is a^3, that is,

$$a^{\frac{3}{4}} = \sqrt[4]{a^3}.$$

And in general $a^{\frac{m}{n}}$ is one of the n equal factors whose product is a^m.

Hence the defining equation

$$a^{\frac{m}{n}} = \sqrt[n]{a^m},$$

as in Art. 292.

338. *What should be the meaning of a negative fraction used as an exponent?*

By Art. 336, the quantity $a^{-\frac{m}{n}}$ is the reciprocal of $a^{\frac{m}{n}}$. Hence

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}} = \sqrt[n]{\frac{1}{a^m}} = \sqrt[n]{a^{-m}}.$$

339. With the meanings now assigned, the rules of operation expressed by the formulas

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(A\,x^p\,y^q \dots)^m = A^m\,x^{mp}\,y^{mq} \dots$$

$$\sqrt[n]{A\,x^p\,y^q \dots} = \sqrt[n]{A}\,x^{\frac{p}{n}}\,y^{\frac{q}{n}} \dots$$

may be applied to all quantities, whatever the values of their exponents, and the results can be interpreted.

340. Illustrative Example. Divide $3\,a^2 b$ by $4\,a\,x\,y$.

$$\begin{aligned} 3\,a^2 b \div 4\,a\,x\,y &= 3\,a^2 b^1 x^0 y^0 \div 4\,a^1 b^0 x^1 y^1 \\ &= \tfrac{3}{4}\,a^{2-1} b^{1-0} x^{0-1} y^{0-1} \\ &= \tfrac{3}{4}\,a\,b\,x^{-1} y^{-1}. \end{aligned}$$

This example, which is the same as Ex. V., Art. 83, is here solved in a manner to show how all examples of division of monomials may be brought under the rule of Art. 84, even though the divisor contain factors not found in the dividend.

341. **Exercises.**

What is the value

1. Of $(\frac{1}{3})^{-2}$?
2. Of $(3\frac{3}{4})^{-3}$?
3. Of $\dfrac{1}{3^{-3}}$?
4. Of $1 : (0.25)^{-4}$?
5. Of $(0.2)^{-3} : 3$?
6. Of $[(3)^2]^3$?
7. Of $[(\frac{3}{4})^{-2}]^3$?
8. Of $[(\frac{1}{2})^3]^{-2}$?
9. Of $(2^{-2})^{-3}$?
10. Multiply a^3 by a^{-2}.
11. Divide a^3 by a^{-2}.
12. Multiply a^3 by a^{-5}.
13. Multiply x^{-2} by x^{-3}.
14. Divide x^{-4} by x^{-6}.
15. Multiply $3x^{-3}$ by $-4x^4$.
16. Divide $3a^2x$ by $\frac{1}{4}ay^2$.
17. Divide $8x^p$ by $\frac{1}{2}x^{-q}$.
18. Divide 1 by a^3x^2.
19. Divide 1 by a^3x^{-2}.
20. Divide 1 by $\frac{1}{2}y^k$.
21. Multiply $3a$ by $\frac{1}{3}a^{-2}$.
22. Divide $\frac{2}{15}k^{-2}$ by $\frac{3}{5}k^{-4}$.
23. Multiply $3y^{-5}$ by $5y^{-3}$.
24. Multiply $-3z^3$ by $-3z^{-3}$.
25. Multiply $-5w^{-2}$ by $5w^2$.
26. Multiply ax^p by px^{-a}.
27. Multiply k^{-l} by k^{2l}.
28. Divide k^l by k^{-2l}.
29. Divide z^{-m} by z^{-2m}.
30. Divide $-a$ by a^{-1}.
31. Divide b^x by $a^m b^y$.
32. Multiply $4a^{-3}x^2$ by $5a^4x^{-4}y$.
33. Divide $-6x^{-2}y^3$ by $3xy^{-1}z^2$.
34. Divide a^k by a^{-k}. What is the quotient when any quantity is divided by its reciprocal?
35. Show that $\dfrac{a^3y^{-2}}{b^2x^{-3}}$ is equal to $\dfrac{a^3x^3}{b^2y^2}$.
36. Show that $\dfrac{1}{x^3} - \dfrac{3}{x^2y} + \dfrac{3}{xy^2} - \dfrac{1}{y^3}$ is equal to
$$x^{-3} - 3x^{-2}y^{-1} + 3x^{-1}y^{-2} - y^{-3}.$$
37. Change $xy^{-2} - 2x^{\frac{1}{2}}y^{-1}z^{-\frac{1}{2}} + z^{-1}$ to an equivalent form having only positive exponents.
38. Prove, in general, that the value of a fraction is not changed by removing a factor from the numerator to the denominator, or from the denominator to the numerator, provided the sign of its exponent be reversed.

39. Raise $3a^2x^{-3}$ to the 3d power.

40. Find the $-$ 3d power of $3x^2y^{-1}$.

41. What is the value of $81^{\frac{3}{4}}$, or $81^{0.75}$?

42. Of $36^{1.5}$?

43. Of $4^{-3.5}+8^{-2\frac{1}{3}}$?

44. Of $4^{-1.5}+9^{-0.5}$?

45. Of $(16)^{-\frac{3}{4}}$?

46. Of $\dfrac{1}{(125)^{-\frac{2}{3}}}$?

47. Find the value of $\frac{2}{3}a-a^{\frac{2}{3}}$ when $a=729$.

48. Find the value of $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ when $a=8$ and $b=64$.

49. When $a=243$, what is the value of $a^{-0.6}$?

50. Multiply $y^{\frac{1}{2}}$ by $y^{\frac{1}{3}}$.

51. Multiply $a^{\frac{3}{4}}$ by $a^{\frac{2}{5}}$.

52. Divide $a^{\frac{3}{4}}$ by $a^{\frac{2}{5}}$.

53. Multiply $5x^{\frac{3}{4}}y^{-\frac{2}{3}}$ by $4x^{-\frac{3}{4}}y^{\frac{2}{3}}$.

54. Divide $4a^{\frac{2}{3}}b^{\frac{1}{3}}$ by $8a^{\frac{1}{3}}b^{-\frac{1}{3}}$.

55. Multiply $3a^{\frac{2}{3}}b^{-\frac{1}{2}}$ by $-6a^{-\frac{1}{3}}b^{\frac{3}{2}}$.

56. Multiply the nth root of x by the mth root of x.

57. Divide the nth root of x by the mth root of x.

58. Multiply $x^{\frac{r}{s}}$ by $x^{\frac{u}{v}}$.

59. Divide $x^{\frac{r}{s}}$ by $x^{\frac{u}{v}}$.

60. Raise $\frac{1}{2}a^{\frac{2}{3}}b^{\frac{3}{2}}$ to the 6th power.

61. Find the $\frac{2}{3}$ power of $\frac{1}{8}a^{\frac{3}{2}}$.

62. Find the $-\frac{3}{4}$ power of $256x^{\frac{2}{3}}y^{-\frac{4}{3}}$.

63. Raise $x^{\frac{r}{s}}$ to the $\frac{u}{v}$ power.

64. Simplify $(2x^{\frac{2}{3}})^{-3}$.

65. Simplify $(a^{\frac{2}{3}})^{\frac{3}{4}}$.

66. Simplify $(b^{-\frac{1}{2}}y^{\frac{3}{2}})^{-\frac{4}{5}}$.

67. Simplify $\left(x^{\frac{2r}{s}}\right)^{\frac{s}{r}}$.

What is the value

68. Of $7^{\frac{3}{4}}\times7^{\frac{3}{2}}\times7^{\frac{7}{4}}+16^{1\frac{3}{17}}\times16^{\frac{5}{17}}\times16^{\frac{1}{34}}$?

69. Of $16^{\frac{3}{2}}+8^{\frac{2}{3}}+16^{\frac{3}{4}}+125^{\frac{1}{3}}-512^{\frac{5}{3}}+100^{0.5}-81^{0.75}$?

70. If $10^{0.30103}=2$, what is the value of $10^{0.90309}$?

71. If $10^{2.08991}=123$, and $10^{2.65896}=456$, what is the value of $10^{4.74887}$?

342. The use of fractional and negative exponents in the multiplication and division of polynomials may be illustrated as follows:

I. Multiplication.

$$\begin{array}{l}
a^{\frac{3}{2}} + 3ab^{\frac{1}{3}} + 3a^{\frac{1}{2}}b^{\frac{2}{3}} + b \\
\underline{a^{\frac{1}{2}} - b^{\frac{1}{3}}\qquad\qquad\qquad\quad} \\
a^2 + 3a^{\frac{3}{2}}b^{\frac{1}{3}} + 3ab^{\frac{2}{3}} + a^{\frac{1}{2}}b \\
\underline{\qquad\; - a^{\frac{3}{2}}b^{\frac{1}{3}} - 3ab^{\frac{2}{3}} - 3a^{\frac{1}{2}}b - b^{\frac{4}{3}}} \\
a^2 + 2a^{\frac{3}{2}}b^{\frac{1}{3}} \qquad\qquad\quad - 2a^{\frac{1}{2}}b - b^{\frac{4}{3}}
\end{array}$$

II. Division.

$$\begin{array}{r|l}
x^2y^{-\frac{4}{3}} - 2 + x^{-2}y^{\frac{4}{3}} & x^{\frac{1}{2}}y^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{1}{3}} \\
\cline{2-2}
\underline{x^2y^{-\frac{4}{3}} - xy^{-\frac{2}{3}}\qquad\quad} & x^{\frac{3}{2}}y^{-1} + x^{\frac{1}{2}}y^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{1}{3}} - x^{-\frac{3}{2}}y \\
xy^{-\frac{2}{3}} - 2 & \\
\underline{xy^{-\frac{2}{3}} - 1} & \\
-1 \qquad\qquad & \\
\underline{-1 + x^{-1}y^{\frac{2}{3}}} & \\
- x^{-1}y^{\frac{2}{3}} + x^{-2}y^{\frac{4}{3}} & \\
\underline{- x^{-1}y^{\frac{2}{3}} + x^{-2}y^{\frac{4}{3}}} & \\
0 \qquad\quad &
\end{array}$$

343. Exercises.

72. Multiply $a^{-1}x^{-1} - ax$ by $x - a$.
73. Multiply $ab^{-1} + 1 + a^{-1}b$ by $ab^{-1} - 1 + a^{-1}b$.
74. Find the 2d power of $ab^{-1} + a^{-1}b$.
75. Find the 3d power of $xy^{-1} - x^{-1}y$.
76. Divide $x^4 - 2 + x^{-4}$ by $x + x^{-1}$.
77. Divide $x^2y^{-2} + 1 + x^{-2}y^2$ by $xy^{-1} + 1 + x^{-1}y$.
78. Divide $a^{-1} - x^{-1} - ax^2 + a^2x$ by $x - a$.
79. Find the square root of
$$x^2y^{-2} + 2xy^{-1} + 3 + 2x^{-1}y + x^{-2}y^2.$$
80. Multiply $a + a^{\frac{1}{2}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{3}}$.
81. Multiply $a - a^{\frac{1}{2}}x^{\frac{1}{2}} + x$ by $a^{\frac{1}{2}} + x^{\frac{1}{2}}$.

82. Multiply $a - a^{\frac{1}{2}}x^{\frac{1}{2}} + x$ by $a + a^{\frac{1}{2}}x^{\frac{1}{2}} + x$.

83. Multiply $a^{\frac{3}{2}} - a^{\frac{2}{3}} + a^{-\frac{1}{3}} - a^{-\frac{3}{2}}$
by $a^{\frac{3}{2}} + a^{\frac{2}{3}} - a^{-\frac{1}{3}} - a^{-\frac{3}{2}}$.

84. Divide $x - y$ by $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.

85. Divide $a + b$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}}$.

86. Divide $m - n$ by $m^{\frac{1}{4}} + n^{\frac{1}{4}}$.

87. Divide $k^2 + l^2$ by $k^{\frac{2}{3}} + l^{\frac{2}{3}}$.

88. Divide $a^6 - b^6$ by $a^{1.5} + b^{1.5}$.

89. Divide $a - b^2$ by $a^{\frac{1}{4}} - b^{\frac{1}{2}}$.

90. Find the 5th root of

$$a^2 - 5\,a^{4.1} + 10\,a^{6.2} - 10\,a^{8.3} + 5\,a^{10.4} - a^{12.5}.$$

91. Find the cube root of

$$a^{-\frac{3}{2}}y^{\frac{3}{2}} - 3\,a^{-1}y + 6\,a^{-\frac{1}{2}}y^{\frac{1}{2}} - 7 + 6\,a^{\frac{1}{2}}y^{-\frac{1}{2}} - 3\,a\,y^{-1} + a^{\frac{3}{2}}y^{-\frac{3}{2}}.$$

344. A root indicated by a radical sign or a fractional exponent is called a ***radical quantity,*** or briefly, a ***radical.***

345. The ***degree*** of a radical is indicated by its index or by the denominator of its fractional exponent.

Thus $\sqrt[3]{a^4}$, $x^{\frac{1}{3}}$, and $(a + x)^{\frac{2}{3}}$, are radicals of the third degree.

346. A radical quantity is ***rational*** when the quantity expressed under the radical sign (or fractional exponent) is a perfect power of the same degree as the root indicated: otherwise it is an ***irrational quantity,*** or a ***surd.***

Thus $\sqrt[3]{27}$, $\sqrt[4]{b^4y^{12}}$, and $8^{-\frac{1}{3}}$, are rational; but $\sqrt{3}$, $\sqrt[3]{20}$, $\sqrt[4]{b^8y^9}$, and $8^{-\frac{1}{2}}$, are irrational quantities, or surds.

347. A radical quantity is reduced to its simplest form by taking out from under the radical sign all the factors that can be taken out. See Art. 297.

Thus the simplest forms of $\sqrt{98}$, $(108)^{\frac{1}{3}}$, and $\sqrt[4]{b^8y^9}$,
are respectively $7\sqrt{2}$, $3(4)^{\frac{1}{3}}$, and $y^2\sqrt[4]{b^8y}$.

348. The quantity written immediately before the radical sign is the ***coefficient of the radical.***

349. Radicals are similar if they are of the same degree, and the quantity expressed under the radical sign is the same in all.

Thus $2\sqrt[3]{4}$ and $5\sqrt[3]{4}$ are similar; but $2\sqrt{5}$ and $2\sqrt[3]{5}$ are not similar; neither are $5\sqrt{2}$ and $5\sqrt{3}$.

Radicals sometimes become similar by reduction. Thus $\sqrt{8}$ and $\sqrt{50}$ when reduced to their simplest forms, $2\sqrt{2}$ and $5\sqrt{2}$, are similar.

350. Most of the operations with radicals are simplest if fractional exponents are used instead of radical signs. This is illustrated by the following examples, where the form of expression is changed whenever the change makes the operation simpler.

351. ILLUSTRATIVE EXAMPLES. I. Add $\sqrt{500}$, $\sqrt{180}$, and $\sqrt{320}$.

The radicals when reduced to their simplest forms are found to be similar, and their sum is found by adding the coefficients and writing the result before the common radical factor.

$$\begin{aligned} \sqrt{500} &= 10\sqrt{5} \\ \sqrt{180} &= 6\sqrt{5} \\ \sqrt{320} &= \underline{8\sqrt{5}} \\ \text{Sum} &= 24\sqrt{5}. \end{aligned}$$

II. From $\sqrt{50}$ subtract $\sqrt{20}$.

The radicals being dissimilar, their difference can only be indicated.

$$\begin{aligned} \sqrt{50} &= 5\sqrt{2} \\ \sqrt{20} &= 2\sqrt{5} \\ \text{Difference} &= 5\sqrt{2} - 2\sqrt{5}. \end{aligned}$$

III. Multiply $4\sqrt{a^3x}$ by $7\sqrt[3]{a^4x^2}$.

By changing the form of expression, the multiplication is performed by adding the exponents. See Arts. 81 and 339.

$$\begin{aligned} 4\sqrt{a^3x} \times 7\sqrt[3]{a^4x^2} &= 4a^{\frac{3}{2}}x^{\frac{1}{2}} \times 7a^{\frac{4}{3}}x^{\frac{2}{3}} \\ &= 28a^{\frac{3}{2}+\frac{4}{3}}x^{\frac{1}{2}+\frac{2}{3}} \\ &= 28a^{\frac{17}{6}}x^{\frac{7}{6}} \\ &= 28\sqrt[6]{a^{17}x^7} \end{aligned}$$

The product may then be changed back to the other form, if desirable.

IV. Divide $6\sqrt[3]{by^2}$ by $2\sqrt[4]{by}$, and $5\sqrt[3]{a^4}$ by $2\sqrt{a^5}$.

By changing the form of expression, the division is performed by subtracting exponents. See Arts. 84 and 339.

The results may then be changed back to the other form.

$$\begin{aligned}6\sqrt[3]{by^2}\div 2\sqrt[4]{by} &= 6b^{\frac13}y^{\frac23}\div 2b^{\frac14}y^{\frac14}\\ &= 3b^{\frac13-\frac14}y^{\frac23-\frac14}\\ &= 3b^{\frac1{12}}y^{\frac5{12}}\\ &= 3\sqrt[12]{by^5}\end{aligned}$$

$$\begin{aligned}5\sqrt[3]{a^4}\div 2\sqrt{a^5} &= 5a^{\frac43}\div 2a^{\frac52}\\ &= \tfrac52 a^{\frac43-\frac52}\\ &= \tfrac52 a^{-\frac76}\\ &= \frac{5}{2a^{\frac76}}=\frac{5}{2\sqrt[6]{a^7}}\end{aligned}$$

V. Raise $2\sqrt[3]{a^2x^4}$ to the 5th power.

See Arts. 245 and 339.

$$\begin{aligned}(2\sqrt[3]{a^2x^4})^5 &= (2a^{\frac23}x^{\frac43})^5\\ &= 2^5a^{\frac{10}{3}}x^{\frac{20}{3}}\\ &= 32\sqrt[3]{a^{10}x^{20}}\end{aligned}$$

VI. Find the square root of $2\sqrt[3]{y^4z^5}$.

This example may be regarded as an application of Art. 245, the value of m being $\frac12$, or as an application of Art. 291, the value of n being 2.

$$\begin{aligned}\sqrt{2\sqrt[3]{y^4z^5}} &= (2y^{\frac43}z^{\frac53})^{\frac12}\\ &= 2^{\frac12}y^{\frac43\times\frac12}z^{\frac53\times\frac12}\\ &= 2^{\frac36}y^{\frac46}z^{\frac56}\\ &= \sqrt[6]{8y^4z^5}\end{aligned}$$

352. Exercises.

92. Add $\sqrt{175}$, $\sqrt{343}$, and $\sqrt{63}$.

93. Add $\sqrt{32}$, $\sqrt{27}$, and $\sqrt{20}$.

94. From $\sqrt{243}$ subtract $\sqrt{27}$.

95. Add $\sqrt{a}$ and $\sqrt{b}$.

96. Add $\sqrt{a}$ and $\sqrt[3]{a}$.

97. Add $2\sqrt{a}$ and $3\sqrt{a}$.

98. Multiply $\sqrt{a}$ by $\sqrt[3]{a}$.

99. Multiply $2\sqrt{a}$ by $3\sqrt{a}$.

100. Multiply $\sqrt{5}$ by $\sqrt[3]{5}$.

101. Divide $\sqrt{5}$ by $\sqrt[3]{5}$.

102. Divide $3\sqrt{2}$ by $2\sqrt[3]{2}$.

103. Multiply $\sqrt[3]{y^5}$ by $\sqrt[4]{y^8}$.

104. Divide $\sqrt[3]{y^5}$ by $\sqrt[4]{y^3}$.

105. Multiply $\sqrt[m]{x^p}$ by $\sqrt[n]{x^q}$.

106. Divide $\sqrt[m]{x^p}$ by $\sqrt[n]{x^q}$.

107. Multiply $\sqrt[3]{a b x^2}$ by $\sqrt[3]{a^2 b y^4}$.

108. Multiply $3 a^2 \sqrt{a m^3}$ by $5 a \sqrt[3]{a m^2}$.

109. Multiply $4 \sqrt{m^3 y^5}$ by $5 \sqrt[3]{m^2 y^4}$.

110. Multiply $2 \sqrt{5 k^3 l^3}$ by $\sqrt{20 k l}$.

111. Multiply y^2 by $\dfrac{1}{\sqrt[5]{y^3}}$.

112. Expand $(\sqrt[4]{a^2} - \sqrt[2]{a^3})^3$.

113. Expand $(\sqrt[m]{a} - \sqrt[n]{b})^3$.

114. Expand $(\sqrt[n]{a} - \sqrt[n]{b})^2$.

115. Expand $(\sqrt[m]{a^n} + \sqrt[n]{a^m})^2$.

116. Raise $\sqrt{a^3}$ to the $-\frac{2}{3}$ power, and reduce the result to its simplest form.

Simplify the following expressions:

117. $(\sqrt[4]{x^3})^{\frac{2}{3}}$.

118. $(\sqrt[4]{a b^2})^{-\frac{4}{5}}$.

119. $\sqrt[3]{a \sqrt{a}}$.

120. $\sqrt[5]{a^2 \sqrt{a}}$.

121. $\sqrt[3]{5 \sqrt[4]{7}}$.

122. $\sqrt{3 \sqrt[3]{5}}$.

123. Simplify $\sqrt[2]{\dfrac{a}{\sqrt[3]{a}}}$, or, in general, $\sqrt[n-1]{\dfrac{a}{\sqrt[n]{a}}}$.

Simplify and find the value

124. Of $(\sqrt[3]{2^5})^{\frac{3}{5}}$.

125. Of $\left(\sqrt{\sqrt[5]{2\frac{14}{25}}}\right)^5$.

126. Of $(\sqrt[3]{3} - \sqrt[3]{2})^3$.

127. Of $(\sqrt{5} - \sqrt{3})^3$.

353. The following examples show how some fractions having denominators partly or wholly irrational can be changed to equivalent fractions with rational denominators. This change is made to avoid the use of an irrational divisor in computing the numerical values of such fractions.

Illustrative Examples. I. Express $\dfrac{2}{\sqrt[3]{5}}$ with a rational denominator.

$$\frac{2}{\sqrt[3]{5}} = \sqrt[3]{\frac{8}{5}} = \sqrt[3]{\frac{8 \times 25}{5 \times 25}} = \sqrt[3]{\frac{200}{125}} = \tfrac{1}{5}\sqrt[3]{200}.$$

II. Express $\dfrac{a}{\sqrt[n]{b}}$ with a rational denominator.

$$\frac{a}{\sqrt[n]{b}} = \sqrt[n]{\frac{a^n}{b}} = \sqrt[n]{\frac{a^n b^{n-1}}{b^n}} = \tfrac{1}{b}\sqrt[n]{a^n b^{n-1}}.$$

III. Express $\frac{5}{3-\sqrt{2}}$ with a rational denominator.

$$\frac{5}{3-\sqrt{2}}=\frac{5}{3-\sqrt{2}}\times\frac{3+\sqrt{2}}{3+\sqrt{2}}=\frac{15+5\sqrt{2}}{7}.$$

IV. Express $\frac{5}{\sqrt{5}+\sqrt{3}}$ with a rational denominator.

$$\frac{5}{\sqrt{5}+\sqrt{3}}=\frac{5}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{5\sqrt{5}-5\sqrt{3}}{2}.$$

The last two examples may be generalized thus:

(III.) Express $\frac{a}{b\pm\sqrt{c}}$ with a rational denominator.

If both numerator and denominator be multiplied by $b\mp\sqrt{c}$, the denominator of the result will be $(b\pm\sqrt{c})(b\mp\sqrt{c})$, or b^2-c, a rational quantity; and the required fraction is $\frac{a(b\mp\sqrt{c})}{b^2-c}$.

(IV.) Express $\frac{a}{\sqrt{b}\pm\sqrt{c}}$ with a rational denominator.

If both numerator and denominator be multiplied by $\sqrt{b}\mp\sqrt{c}$, the denominator of the result is $(\sqrt{b}\pm\sqrt{c})(\sqrt{b}\mp\sqrt{c})$, or $b-c$, a rational quantity; and the required fraction is $\frac{a(\sqrt{b}\mp\sqrt{c})}{b-c}$.

354. Exercises.

Free the following fractions from irrational denominators,

128. $\frac{4}{2\sqrt[3]{4}}$.

129. $\frac{3}{2\sqrt[3]{10}}$.

130. $\frac{3}{\sqrt[3]{5}}$.

131. $\frac{1}{4\sqrt[3]{2}}$.

132. $2\sqrt{\frac{1}{2}}$.

133. $3\sqrt[3]{\frac{1}{3}}$.

134. $\frac{2}{\sqrt{3}+\sqrt{2}}$.

135. $\frac{3}{\sqrt{5}+\sqrt{2}}$.

136. $\frac{4}{\sqrt{5}-1}$.

137. $\dfrac{1}{2\sqrt{2}-\sqrt{3}}$.

138. $\dfrac{2+\sqrt{3}}{3+\sqrt{3}}$.

139. $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$.

140. $\dfrac{1}{\sqrt{2-\sqrt{3}}}$.

141. $\dfrac{2+\sqrt{2}}{\sqrt{2-\sqrt{2}}}$.

142. $\dfrac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$.

143. $\dfrac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}$.

144. $\dfrac{1}{a-\sqrt{a^2-x^2}}$.

145. $\dfrac{1}{\sqrt[3]{a-\sqrt{a}}}$.

146. $\dfrac{1}{\sqrt{x+1}+\sqrt{x-1}}$.

147. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

355. Equations of the form $Ax^{2n}+Bx^n=M$, in which n has a fractional or negative value, may be solved as quadratics for two values of x^n, from which may be found values of x by the method of Art. 311.

ILLUSTRATIVE EXAMPLES. I. Solve the equation $x^{\frac{5}{3}}-8x^{\frac{5}{6}}=768$.

In this example $n=\frac{5}{6}$ and $2n=\frac{5}{3}$. Solving the equation as a quadratic, we get two values of $x^{\frac{5}{6}}$, and, from these, two values of x.

$$x^{\frac{5}{3}}-8x^{\frac{5}{6}}=768$$
$$x^{\frac{5}{3}}-8x^{\frac{5}{6}}+16=784$$
$$x^{\frac{5}{6}}=4\pm 28$$
$$x_1^{\frac{5}{6}}=32 \qquad x_2^{\frac{5}{6}}=-24$$
$$x_1^{\frac{1}{6}}=2 \qquad x_2^{\frac{1}{6}}=(-24)^{\frac{1}{5}}$$
$$x_1=64 \qquad x_2=(-24)^{\frac{6}{5}}$$

II. Solve the equation $4\sqrt[4]{x}+\sqrt{x}=21$.

Writing the equation with fractional exponents, we find it to be of the quadratic form, and solve it with reference to $x^{\frac{1}{4}}$.

$$x^{\frac{1}{2}}+4x^{\frac{1}{4}}=21$$
$$x^{\frac{1}{2}}+4x^{\frac{1}{4}}+4=25$$
$$x^{\frac{1}{4}}=-2\pm 5$$
$$=3, \text{ or } -7.$$
$$x=81, \text{ or } 2401.$$

356. Exercises.

148. $x+\sqrt{x}=30$.

149. $x-\sqrt{x}=30$.

150. $5x=39+2\sqrt{x}$.

151. $\sqrt[4]{x}+\sqrt{x}=20$.

152. $x^3 - 10\sqrt{x^3} = 459.$

153. $\sqrt[3]{x} + 7\sqrt[3]{x^2} = 350.$

154. $\sqrt[3]{x^2} + \sqrt[3]{x} = n(n+1).$

155. $\dfrac{\sqrt{x}}{21-\sqrt{x}} + \dfrac{21-\sqrt{x}}{\sqrt{x}} = 2\frac{1}{2}.$

156. $\dfrac{x-\sqrt{x}}{x+\sqrt{x}} = \dfrac{36}{x^2-x}.$

157. $\dfrac{28+\sqrt{x}}{28-\sqrt{x}} = \dfrac{9+3\sqrt{x}}{9+2\sqrt{x}}.$

158. $x + x^{-1} = 2\frac{1}{6}.$

NOTE. Multiply each member by x.

159. $x^2 - x^{-2} = 6.09.$

160. $\sqrt{x^2} + \sqrt{x^3} = 6\sqrt{x}.$

161. $8x^{-3} + 5x^{-\frac{3}{2}} = 108.$

162. $12x^{-\frac{3}{4}} - x^{-\frac{3}{8}} = 2^{-4}.$

163. $x^{\frac{m}{n}} - ax^{\frac{m}{2n}} = b.$

357. When the quantity under the radical sign consists of two or more terms one or more of which are unknown, the equation may be cleared of radicals and solved in the way shown by the following examples.

ILLUSTRATIVE EXAMPLES. I. Solve the equation $x \pm \sqrt{25-x^2} = 1.$

We first reduce the equation to a form in which the radical quantity stands alone as one member of the equation. Then we square each member, and have an equation which solved gives two values of x.

$$\begin{aligned} x \pm \sqrt{25-x^2} &= 1 \\ \pm\sqrt{25-x^2} &= 1-x. \\ 25-x^2 &= 1-2x+x^2. \\ 2x^2-2x &= 24 \\ x^2-x &= 12 \\ x &= 4, \text{ or } -3. \end{aligned}$$

NOTE. It is to be observed that the one value of x satisfies the equation when the radical is taken with the sign +, and the other when it is taken with the sign −.

II. Solve the equation $\sqrt{9x+4} - \sqrt{3x+1} = \sqrt{2x-1}.$

$$\begin{aligned} \sqrt{9x+4} - \sqrt{3x+1} &= \sqrt{2x-1} \\ 9x+4-2\sqrt{9x+4}\sqrt{3x+1}+3x+1 &= 2x-1 \\ \sqrt{9x+4}\sqrt{3x+1} &= 5x+3 \\ (9x+4)(3x+1) &= (5x+3)^2 \\ 27x^2+21x+4 &= 25x^2+30x+9 \\ 2x^2-9x &= 5 \\ x &= 5, \text{ or } -\tfrac{1}{2} \end{aligned}$$

We square each member, reduce the result so that the product of the two radicals stands alone as one member, square again, and have an equation free from radicals, giving, when reduced and solved, two values of x.

358. Exercises.

164. $x+2-\sqrt{16+x^2}=0.$
165. $x=9-\sqrt{x^2+9}.$
166. $x+\sqrt{25+x}=157.$
167. $x+1-\sqrt[3]{x^3+19}=0.$
168. $3+\sqrt[3]{x^3-9x}=x.$
169. $\sqrt{12+x}=2+\sqrt{x}.$
170. $\sqrt{4+x}=\sqrt[4]{x^2+20x+9}.$
171. $\sqrt{3-x}-\sqrt[4]{3+x^2}=0.$
172. $\sqrt{x}+\sqrt{x+15}=15.$
173. $2+\sqrt{3x}=\sqrt{5x+4}.$
174. $\sqrt{x+11}+\sqrt{x-4}=5.$
175. $\dfrac{1}{\sqrt{x-1}}-\dfrac{1}{\sqrt{x+1}}=\dfrac{1}{\sqrt{x^2-1}}.$
176. $\sqrt{x^2+2x}+\sqrt{x^2-2x}=2\sqrt{5}.$
177. $\sqrt{2x-27a}=9\sqrt{a}-\sqrt{2x}.$
178. $\sqrt{4+x}-\sqrt{1+x}=2\sqrt{x-2}.$
179. $\sqrt{8+x}-\sqrt{x}=2\sqrt{1+x}.$
180. $\sqrt{2x-45}=3\sqrt{15}-\sqrt{2x}.$
181. $\sqrt{x}+\sqrt{4+x}=\dfrac{2}{\sqrt{x}}.$
182. $\sqrt{x}+\sqrt{9+x}=\dfrac{45}{\sqrt{9+x}}.$
183. $\sqrt{2x+2}+\sqrt{7+6x}=\sqrt{7x+72}.$
184. $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}.$
185. $\sqrt{(x-a)(x-b)}+(\sqrt{(x-c)(x-a+b-c)}=\sqrt{(a-c)(b-c)}.$
186. $\sqrt{(z-a)^2+2ab+b^2}=z-a+b.$
187. $\sqrt{(z+a)^2+2ab+b^2}=b-a-z.$
188. $\sqrt{x}-\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}.$
189. $(x-1)^{\frac{1}{2}}=x+\frac{5}{4}.$
190. $(4+y)^{\frac{1}{2}}+(5+y)^{\frac{1}{2}}=9.$
191. $z^{\frac{1}{2}}+(z-4)^{\frac{1}{2}}=8(z-4)^{-\frac{1}{2}}.$
192. $6+(x^2-1)^{\frac{1}{2}}=16(x^2-1)^{-\frac{1}{2}}.$

SECTION XII.

PROGRESSIONS.

Arithmetical Progression.

359. A series of quantities so arranged that each term is greater than the one before it by a constant quantity is an ***arithmetical progression.***

Thus $3,\ 5,\ 7,\ 9,\ 11\ \ldots$

$m,\ m+x,\ m+2x,\ m+3x\ \ldots$

are examples of arithmetical progression. In the first, each term exceeds the one before it by 2, and in the second by x.

The word *greater* in this definition is to be taken in its algebraic sense, so that the excess of each term over the one before it may be in some cases a constant *negative* quantity. Hence the definition includes such series as the following:

$$9,\ 5,\ 1,\ -3,\ -7,\ -9,\ -13\ \ldots$$

$$b+2y,\ b+y,\ b,\ b-y,\ b-2y\ \ldots$$

where the *excess* of each term over the one before it is -4 in the first, and $-y$ in the second.

360. The constant quantity by which each term of an arithmetical progression exceeds the one before it is called the ***common difference.***

NOTE. An arithmetical progression is sometimes called a *progression by difference.*

361. Each term after the first term of an arithmetical progression is found by adding the common difference to the preceding term. If the common difference is positive, the progression is said to be an *increasing*, if negative, a *decreasing* progression.

362. In the treatment of arithmetical progression the following notation will be used:

$a =$ the first term.
$l =$ the last term.
$n =$ the number of terms.
$d =$ the common difference.
$S =$ the sum of all the terms.

With this notation an arithmetical progression is thus represented:

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d\ \ldots$$

363. *To find an expression for the* nth, *or last, term.*

In the series above given, the coefficient of d is 1 in the second term, 2 in the third term, 3 in the fourth term, and so on, being always one less than the number of the term. Accordingly in the n^{th} term the coefficient of d is $n-1$, and the n^{th} term is equal to $a+(n-1)d$. Hence the formula

$$l = a+(n-1)d.$$

364. *To find an expression for the sum of all the terms.*

By the definition of S,

$$S = a+[a+d]+[a+2d]+\ \ldots\ +[a+(n-1)d].$$

If the series be written in the reverse order, beginning with l, the next term will be $l-d$, the next, $l-2d$, and so on to the original first term, which will be $l-(n-1)d$. Hence we have as another expression for S,

$$S = l+[l-d]+[l-2d]+\ \ldots\ +[l-(n-1)d].$$

By adding these two values of S term by term we get,

$$2S = [a+l]+[a+l]+[a+l]+\ \ldots\ +[a+l].$$

In the second member of this equation the quantity $[a+l]$ occurs as many times as there are terms in the given series, that is n times. Hence

$$2S = (a+l)n$$
$$S = \tfrac{1}{2}(a+l)n,$$

or, in words,

The sum of all the terms of an arithmetical progression is equal to half the sum of the two extreme terms multiplied by the number of terms.

365. The two formulas

$$l = a + (n-1)d,$$

and
$$S = \tfrac{1}{2}(a+l)n$$

contain all five elements (a, l, n, d, S) of an arithmetical progression. They are independent simultaneous equations. If, therefore, any three of the five elements are given, the other two may be found by solving these equations. We have only to substitute the known elements for their symbols in the equations, and then solve them by methods already explained.

366. Exercises.

1. Find l and S, when $a = 1$, $n = 100$, and $d = 3$.
2. Find n and S, when $a = 40$, $l = 4$, and $d = -3$.
3. Find a and S, when $l = 30$, $n = 16$, and $d = 3$.
4. Find n and d, when $a = 20$, $l = -8$, and $S = 48$.
5. Find l and d, when $a = 40$, $n = 12$, and $S = 150$.
6. Find a and d, when $l = 20$, $n = 9$, and $S = 36$.
7. Find a and l, when $n = 10$, $d = 4$, and $S = 1000$.
8. Find d and S, when $a = 30$, $l = 15$, and $n = 7$.
9. Find l and n, when $a = 9$, $d = 3$, and $S = 306$.

NOTE. The negative value of n resulting from the quadratic equation can have no meaning, and must be rejected.

10. Find a and n when $l = 37$, $d = 4$, and $S = 190$.

NOTE. The fractional value of n resulting from the quadratic equation can have no meaning, and must be rejected.

367. Problems in Arithmetical Progression.

11. Find the 42d term of the series 4, $6\frac{1}{2}$, 9 . . .
12. Find the 40th term of the series -20, -18, -16 . . .

Find the sum,

13. Of all the integers from 1 to 100 inclusive.
14. Of all the integers from 1 to 1000 inclusive.
15. Of all the odd numbers from 1 to 99 inclusive.
16. Of all the even numbers from 2 to 100 inclusive.
17. Of the series -10, -7, -4, . . . to 20 terms.

18. Find the number of terms, when the first term is 17, the common difference -2, and the sum of all the terms 72.

19. How many terms of the series $21 + 18 + 15 + \ldots$ amount to 81? What is the last term?

20. How many terms of the series $19 + 18 + 17 + \ldots$ amount to 124?

21. A contract for boring an artesian well provided that 25 cents should be paid for the first foot, and half a cent more for each succeeding foot, however deep the well might be. The well was sunk to a depth of 1500 feet. How much was due by the contract?

22. It is a law of falling bodies, that, when there is no resistance of the air, the space fallen through in the first second is $16\frac{1}{12}$ feet, in the next, $48\frac{1}{4}$, in the next, $80\frac{5}{12}$, and so on in arithmetical progression. How many feet does a body fall through in t seconds? How many when $t = 20$?

23. In how many seconds will a body starting from a state of rest fall through 4116 feet?

24. Find the n^{th} term of the series $1, 3, 5, 7 \ldots$ and of the series $2, 4, 6, 8 \ldots$

25. Find the sum of $1 + 2 + 3 + 4 + \ldots$ to n terms.

26. Find the sum of $1 + 3 + 5 + 7 + \ldots$ to n terms.

27. Find the sum of $2 + 4 + 6 + 8 + \ldots$ to n terms.

28. Find the sum of $\frac{1}{4} + \frac{3}{8} + \frac{1}{2} + \ldots$ to n terms.

29. Find the sum of $\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \ldots$ to n terms.

30. Find the sum of an arithmetical progression of which the 5th term is 14, the 8th term 23, and the number of terms 12.

31. The 7th term of an arithmetical progression is -6, the 37th, $15\frac{3}{4}$, and the whole number of terms is 55. Find the first and last terms, the common difference, and the sum of all the terms.

32. The sum of the 19th, 43d, and 57th terms of an arithmetical progression is 827. The sum of the 27th, 58th,

69th, and 73d, is 1581. Find the first term and the common difference.

33. The prizes for a boat-race were so fixed that each crew received $15 less than the crew next ahead of it. The leading crew received $120; all the others together, $330. How many prizes were there?

34. The sum of the first nine terms of an arithmetical progression is 117, the sum of the last five is 110, and there are ten terms in all. Find the first term and the common difference.

35. Eight numbers are in arithmetical progression. The sum of the two middle terms is 29, and the product of the first and last terms is 100. Find the first term and the common difference.

36. A and B start at the same time from two towns 165 miles apart, intending to meet on the way. A goes 1 mile the first day, 2 the second, 3 the third, and so on. B goes 20 miles the first day, 18 the second, 16 the third, and so on. After how many days will they meet?

37. Prove that the average value of the terms of an arithmetical progression is $\frac{1}{2}(a+l)$.

38. Prove that the sum of the $(k-m)^{\text{th}}$ and the $(k+m)^{\text{th}}$ terms of an arithmetical progression is equal to twice the k^{th} term.

The first and last terms of a progression are called the extremes, the rest are called means. If it is required to insert a given number of arithmetical means between given extremes, we have merely to find the common difference from the given values of a, l, and n (n being greater by 2 than the number of means to be inserted), and then write out the whole series.

39. Insert 27 arithmetical means between 3 and 17.

40. Insert 10 arithmetical means between 10 and 20.

41. It is required to insert m arithmetical means between h and k. What must be the common difference?

42. The arithmetical mean between two quantities is a quantity as much greater than the one as it is less than the other. Prove that the arithmetical mean between two quantities is equal to half their sum.

Geometrical Progression.

368. A series of quantities so arranged that each term bears a fixed ratio to the one before it is a ***geometrical progression.***

Thus $3, 6, 12, 24, 48 \dots$
$m, mx, mx^2, mx^3 \dots$
are examples of geometrical progression. In the first the ratio of each term to the one before it is 2, and in the second, x.

369. The fixed ratio which each term bears to the one before it is called the ***common ratio.***

NOTE. The common ratio is also called the common quotient; and a geometrical progression is sometimes called a *progression by quotient.*

370. Each term of a geometrical progression is found by multiplying the preceding term by the common ratio. If the common ratio is negative, the terms of the progression are alternately positive and negative.

Thus $-5, +15, -45, +135 \dots$
where the common ratio is -3,
and $+4, -2, +1, -\frac{1}{2}, +\frac{1}{4}, -\frac{1}{8} \dots$
where the common ratio is $-\frac{1}{2}$,
have terms alternately positive and negative.

If the common ratio is numerically less than 1 (that is, if it is a positive or negative proper fraction), the progression is said to be ***convergent;*** if numerically equal to or greater than 1, the progression is said to be ***divergent.***

Thus $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81} \dots$
where the common ratio is $\frac{1}{3}$,
and $2, -\frac{4}{5}, \frac{8}{25}, -\frac{16}{125}, \frac{32}{625} \dots$
where the common ratio is $-\frac{2}{5}$,
are convergent.

371. In the treatment of geometrical progression the following notation will be used:

$a =$ the first term.
$l =$ the last term.
$n =$ the number of terms.
$r =$ the constant ratio.
$S =$ the sum of all the terms.

With this notation a geometrical progression is thus represented:

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5\ \ldots$$

372. *To find an expression for the* nth, *or last, term.*

In the series above given, the exponent of r is 1 in the second term, 2 in the third term, 3 in the fourth term, and so on, being always one less than the number of the term. Accordingly in the n^{th} term the exponent of r is $n-1$, and the n^{th} term is equal to ar^{n-1}. Hence the formula

$$l = ar^{n-1}.$$

373. *To find an expression for the sum of all the terms.*

By definition of S,

$$S = a + ar + ar^2 + \ldots + ar^{n-1}.$$

Multiplying each member of this equation by r,

$$rS = ar + ar^2 + ar^3 + \ldots + ar^n.$$

Subtracting the first equation from the second, omitting terms which cancel,

$$rS - S = ar^n - a,$$

whence the formula

$$S = \frac{a(r^n - 1)}{r - 1}.$$

374. The two formulas

$$l = ar^{n-1}$$

$$S = \frac{a(r^n - 1)}{r - 1}$$

containing all five elements of a geometrical progression, are independent simultaneous equations, from which may be found any two elements when the other three are given.

In some cases, however, the equations cannot be solved by the methods heretofore given. Thus, when n is unknown, the equation for finding n is of a kind called exponential equations, the solution of which usually requires the use of logarithms, although in certain simple cases they may be solved by inspection. Also, when r and either a or l are unknown, the equations for finding these elements are of the n^{th} degree, n being the given number of terms. But no general methods have been given for the solution of equations of a higher degree than the second. Nevertheless, the equation for finding a or l is frequently solvable by inspection.

375. Exercises.

43. Find l and S, when $a = 5$, $n = 6$, and $r = -\frac{2}{5}$.

44. Find a and S, when $l = 0.004$, $n = 5$, and $r = 0.1$.

45. Find a and l, when $n = 6$, $r = \frac{1}{3}$, and $S = 7\frac{119}{243}$.

46. Find r and S, when $a = 2\frac{1}{4}$, $l = \frac{8}{27}$, and $n = 6$.

47. Find a and r, when $l = 48$, $n = 5$, and $S = 93$.

SOLUTION.

$$(1)\qquad 48 = ar^4$$

$$(2)\qquad 93 = \frac{ar^5 - a}{r - 1}$$

$$(3)\qquad 93r - 93 = ar^4r - a$$

$$(4)\qquad 93\sqrt[4]{\frac{48}{a}} - 93 = 48\sqrt[4]{\frac{48}{a}} - a$$

$$45\sqrt[4]{\frac{48}{a}} = 93 - a$$

$$(45)^4 \times \frac{48}{a} = (93 - a)^4$$

$$(5)\qquad a(93 - a)^4 = 48 \times 45^4$$

$$(6)\qquad a(93 - a)^4 = 3 \times 90^4$$

$$a = 3$$

$$(7)\qquad 48 = 3r^4$$

$$r^4 = 16$$

$$r = 2$$

By substituting the given elements in the formulas, the results are (1) and (2). Clear (2) of fractions, at the same time writing the term ar^5 in the form ar^4r. This gives (3). Substitute in (3) the value of r derived from (1), which is $\sqrt[4]{\frac{48}{a}}$; also the value of ar^4, which is 48. The result is (4), which reduced gives (5), an equation of the 5th degree, with a as the unknown quantity. This equation would be satisfied by the value $a = 48$. But if $a = 48$, S would equal, not 93, but 240. Hence this value is rejected. By separating 48 into two factors, 3×16, one of which is a perfect fourth power ($16 = 2^4$), the second member of (5) is changed to the form $3 \times 2^4 \times 45^4$, or 3×90^4 as in (6). It is easy to see that (6) is satisfied by the value $a = 3$. This value substituted in (1) gives (7), from which may be found the value of r.

48. Find l and r, when $a = 64$, $n = 6$, and $S = 126$.

SOLUTION.

The method of solution is similar to that of the preceding exercise. Eliminate r by substituting in (2) its value, $\sqrt[5]{\frac{l}{64}}$, derived from (1), at the same time replacing $64r^5$ by l.

$$(1) \qquad l = 64r^5$$

$$(2) \qquad 126 = \frac{64r^5r - 64}{r - 1}$$

$$l(126 - l)^5 = 64 \times 62^5 = 2 \times 124^5$$

$$l = 2.$$

$$r = \sqrt[5]{\tfrac{2}{64}} = \sqrt[5]{\tfrac{1}{32}} = \tfrac{1}{2},$$

49. Find n and S, when $a = 85\frac{1}{3}$, $l = 20\frac{1}{4}$, and $r = \frac{3}{4}$.

SOLUTION.

By substituting the given elements in the formula $l = ar^{n-1}$ we get (1), which reduced gives the exponential equation (2). This may be solved by inspection; for it is easy to see that $\frac{3}{4}$ raised to the 5th power equals $\frac{243}{1024}$.

$$(1) \qquad \tfrac{81}{4} = \tfrac{256}{3}(\tfrac{3}{4})^{n-1}$$

$$(2) \qquad (\tfrac{3}{4})^{n-1} = \tfrac{243}{1024}$$

$$\text{whence} \qquad n - 1 = 5$$

$$n = 6$$

50. Find l and n, when $a = 2$, $r = -3$, and $S = -364$.

SOLUTION. The substitution of the given elements in the formula for S gives the exponential equation $(-3)^n = 729$, which is easily solved by inspection; for since $(-3)^6 = 729$, $n = 6$.

51. Find a and n, when $l = 1458$, $r = 3$, and $S = 2186$.

SOLUTION.

Multiply by 3 each member of (1). The result is (3). Eliminate the quantity $a \times 3^n$ from (2) and (3). The result is (4), from which the value of a is found. The value of n may be found from the exponential equation (3) after substituting therein the value of a.

$$(1) \qquad 1458 = a \times 3^{n-1}$$

$$(2) \qquad 2186 = \frac{a \times 3^n - a}{3 - 1}$$

$$(3) \qquad 4374 = a \times 3^n$$

$$(4) \qquad 2186 = \frac{4374 - a}{3 - 1}$$

$$a = 2$$

52. Find r and n, when $a = 12$, $l = 1\frac{47}{81}$, and $S = 32\frac{68}{81}$.

SOLUTION.

The method of solution is similar to that of the preceding exercise. From (1) is obtained (3), from which and (2) is eliminated the quantity $12r^n$. The result is (4), from which the value of r is found.

$$\frac{128}{81} = 12r^{n-1} \tag{1}$$

$$\frac{2660}{81} = \frac{12r^n - 12}{r - 1} \tag{2}$$

$$\frac{128r}{81} = 12r^n \tag{3}$$

$$\frac{2660}{81} = \frac{\frac{128r}{81} - 12}{r - 1} \tag{4}$$

$$r = \frac{1688}{2532} = \frac{2}{3}$$

376. Problems in Geometrical Progression.

53. Find the common ratio and the sum of six terms of the series $3\frac{3}{8}$, $2\frac{1}{4}$, $1\frac{1}{2}$ . . .

54. Four numbers are in geometrical progression. Their sum is 280, and the smallest one is 7. What is the largest one, and the common ratio?

55. Supposing the population of Chicago to have increased regularly from year to year in geometrical progression, what was the yearly rate of increase from A.D. 1860, when the population was 109,260, to A.D. 1870, when the population was 298,977?

56. From a cask containing 20 gallons of pure brandy, 3 gallons are drawn out and 3 gallons of water poured in. Then the liquid in the cask is weakened a second time in like manner, then a third time, and so on. How much pure brandy remains in the cask after 24 such weakenings?

57. Generalize the foregoing, putting for 20, 3, and 24, the letters a, b, and n respectively.

58. A mixture of silver and copper is made, containing 76 pounds of the former, and 20 pounds of the latter. From this mixture 20 pounds are taken away, and the remainder melted with 20 pounds of copper. Then the new mixture is treated in the same way. When this has been done 24 times, how much pure silver remains in the mixture?

59. Generalize the foregoing, putting for 76, 20, and 24, e letters a, b, and n respectively.

60. There are four numbers in geometrical progression, which being diminished respectively by 7, 3, 2, and 3, are in arithmetical progression. What are the numbers?

61. Find four numbers in arithmetical progression, which, being increased respectively by 5, 7, 11, and 18, will be in geometrical progression.

62. Find an expression in terms of a, r, and n, for the continued product of all the terms of a geometrical progression.

63. Prove that the reciprocals of the terms of a geometrical progression also form a geometrical progression.

64. What are the properties of a series of products formed by multiplying each term of a geometrical progression by the term next following it?

65. Prove that the series of sums or of remainders formed by adding or subtracting each term of a geometrical progression to or from the term next following it is also a geometrical progression. What is its common ratio?

66. Prove that the product of the $(k-m)^{\text{th}}$ and the $(k+m)^{\text{th}}$ terms of a geometric progression is equal to the square of the k^{th} term.

If it is required to insert a given number of geometric means between given extremes, we have merely to find the common ratio from the given values of a, l, and n (n being greater by 2 than the number of means to be inserted), and then write out the whole series.

67. Insert four geometric means between $\frac{8}{27}$ and $2\frac{1}{4}$.

68. Insert eleven geometric means between 1 and 2.

69. It is required to insert m geometric means between h and k. What must be the common ratio?

70. Insert one geometric mean between A and B.

71. Prove that the geometric mean between two quantities is equal to the mean proportional between them (Art. 196).

72. The arithmetical mean between two numbers is 15, and the geometrical mean is 12. What are the numbers?

73. Prove, in general, that two quantities may be found

if their arithmetical and geometrical means are both given. Find expressions for them.

Note. There are some questions relating to compound interest, annuities, sinking-funds, &c., which might be introduced here as depending for their solution on the principles of geometric progression; but the numerical computations are tedious without the use of logarithms. Such questions, therefore, are better placed at the end of the next Section.

The Sum of an Infinite Number of Terms.

377. In a convergent geometrical progression the value of r is a proper fraction. Now, the second power of a proper fraction is numerically less than the first power, the third power is less than the second, and so on. The greater the value given to n, the less becomes the value of r^n; that is, the more nearly does its value approach the limit 0. If we make n greater than any assignable number, that is, $n = \infty$, the value of r^n will become less than any assignable number, that is, $r^n = 0$.

Therefore, for a convergent geometrical progression of an infinite number of terms, the formulas for the last term and sum become

$$l = a r^{n-1} = \frac{a r^n}{r} = \frac{a \times 0}{r} = 0.$$

$$S = \frac{a(r^n - 1)}{r - 1} = \frac{a(0 - 1)}{r - 1} = \frac{-a}{r - 1} = \frac{a}{1 - r}.$$

The formula $$S = \frac{a}{1 - r}$$

gives what is called the sum of all the terms of an infinite convergent geometrical progression. The meaning is, not that all the terms could ever be written down and added, for this would be impossible, but that, the greater the number of terms taken, the more nearly does their sum approach the definite limit $\frac{a}{1 - r}$.

Thus, in the infinite series

$$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\ \cdot\ \cdot\ \cdot\ +\tfrac{1}{1048576}+\ \cdot\ \cdot\ \cdot\ +\tfrac{1}{\infty},$$

the sum of the first three terms is $\frac{7}{8}$, or $1-\frac{1}{8}$
the sum of the first five terms, $\frac{31}{32}$, or $1-\frac{1}{32}$
the sum of the first twenty terms, $\frac{1048575}{1048576}$, or $1-\frac{1}{1048576}$

and so on; the greater the number of terms taken, the more nearly does their sum approach the limit 1, which is the value of S in this case.

$$S=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1.$$

378. Exercises.

74. Find the sum of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ .\ .\ .$ to an infinite number of terms.

75. Find the sum of $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\ .\ .\ .$ to an infinite number of terms.

76. Find the sum of $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ .\ .\ .$ to an infinite number of terms.

77. Find the sum of $\frac{15}{100}+\frac{15}{10000}+\frac{15}{1000000}+\ .\ .\ .$ to an infinite number of terms.

78. Find the sum of $\frac{2}{3}+(\frac{2}{3})^2+(\frac{2}{3})^3+(\frac{2}{3})^4+\ .\ .\ .$ to an infinite number of terms.

79. Find the sum of $18+12+8+\ .\ .\ .$ to an infinite number of terms.

80. Find the sum of $0.3+0.03+0.003+\ .\ .\ .$ to an infinite number of terms.

81. What must be the common ratio that the sum of an infinite geometric progression beginning with $\frac{1}{2}$ may be 2?

82. What is the common ratio of an infinite geometric progression beginning with $\frac{1}{12}$, if the sum of all the terms is $\frac{1}{11}$?

83. Achilles pursues a tortoise with a speed twelve times as great as that of the tortoise, but from a starting-point one stadium behind it. When he comes to the place where the tortoise was at the outset, he finds that the tortoise has gone $\frac{1}{12}$ of a stadium farther on. He runs that distance, and finds the tortoise is still $\frac{1}{144}$ of a stadium ahead of him; he runs

$\frac{1}{144}$ of a stadium, and finds the tortoise still ahead of him $\frac{1}{1728}$ of a stadium; and so the race continues. The question is whether Achilles will ever overtake the tortoise; and, if so, where.

Circulating Decimals.

379. The formula $S = \dfrac{a}{1-r}$ may be used to find, in the form of a common fraction, the exact value of a circulating decimal.

Thus, $0.\dot{2}9\dot{7}$ or $0.297297297\ldots$

may be expressed in the form

$$\tfrac{297}{1000} + \tfrac{297}{1000000} + \tfrac{297}{1000000000} + \cdots$$

an infinite geometric progression in which $a = \frac{297}{1000}$, $r = \frac{1}{1000}$.

Then $S = \dfrac{\frac{297}{1000}}{1 - \frac{1}{1000}} = \frac{0.297}{0.999} = \frac{297}{999} = \frac{11}{37} =$ the exact value of the given circulating decimal.

This may be verified by reducing $\frac{11}{37}$ to a decimal.

If the first part of the decimal does not repeat, the value of the repeating part may be found, and added to the other. Thus

$$0.5\dot{3}\dot{6} \text{ or } 0.5363636\ldots$$

may be expressed in the form

$$\tfrac{5}{10} + \tfrac{36}{1000} + \tfrac{36}{100000} + \tfrac{36}{10000000} + \cdots$$

The sum of all the terms except the first is, by the formula,

$$\frac{\frac{36}{1000}}{1 - \frac{1}{100}} = \frac{0.036}{0.99} = \frac{36}{990} = \frac{2}{55}.$$

Adding this to the first term, we find as the exact value

$$\tfrac{5}{10} + \tfrac{2}{55} = \tfrac{59}{110}.$$

380. Exercises.

Find the value, in the form of a common fraction reduced to its smallest terms,

84. Of $0.111\ldots$
85. Of $0.333\ldots$
86. Of $0.666\ldots$
87. Of $0.5454\ldots$
88. Of $0.00333\ldots$
89. Of $0.0555\ldots$
90. Of $0.58333\ldots$
91. Of $0.\dot{1}8\dot{5}$.
92. Of $2.32\dot{7}$.
93. Of $0.\dot{1}88\dot{1}$.
94. Of $2.43\dot{2}4\dot{3}$.
95. Of $0.291\dot{6}$.
96. Of $0.\dot{4}2857\dot{1}$.

SECTION XIII.

LOGARITHMS.

381. We have seen that powers of the same quantity are multiplied together by adding the exponents, that one power is divided by another by subtracting one exponent from the other, that a power of a power is found by multiplying the exponent, and that a root of a power is found by dividing the exponent. See Arts. 80, 83, 245, 291, 292, and 334–339.

Now, if some one number should be selected as a base, and by means of suitable exponents all numbers should be expressed as powers of this base, then all the labor of multiplying and dividing numbers could be replaced by the mere addition and subtraction of these exponents, and the labor of finding the powers or the roots of numbers could be replaced by the simple process of multiplying or dividing the exponents.

382. A method of abridging the labor of computation by the aid of such a system of exponents was invented by Baron Napier of Scotland, who gave the name *logarithms* to the exponents so used.

This method, first published in 1614, was soon afterwards modified, and, for practical use, much improved, by Henry Briggs, an English mathematician, who adopted for the base of his system the number 10, and with this computed the first table of logarithms of the kind now in common use.

Theory of Logarithms.

383. The ***base*** of a system of logarithms is a chosen number of which all other numbers are regarded as powers.

The base may have any positive value except 1.

The base of the common system of logarithms is 10.

384. The ***logarithm*** of a number is the exponent of the power to which the base must be raised to produce the number.

The abbreviation ***log*** is used for the words *the logarithm of.*

385. The value of x which satisfies the equation

$$b^x = N$$

is the logarithm of N in a system of which the base is b.

386. If we assign to b some fixed positive value other than 1, and let N represent in turn every positive number from 0 to $+\infty$, there will be found, by solving the equation for each value of N, a corresponding value of x. These values of x taken together make up a ***system of logarithms.***

There may be an infinite number of systems of logarithms, because an infinite number of positive values may be assigned to b the base. Only two systems, however, have ever been in use. One, called the Napierean system, has for a base the definite limit to which the value of the expression $\left(1+\frac{1}{m}\right)^m$ approaches as greater and greater values are assigned to m. When m is infinite, the expression becomes $\left(1+\frac{1}{\infty}\right)^\infty = 2.7182818 \ldots$ The system of logarithms having this peculiar number for a base is simpler than any other, and is often called the *natural* system. It is used in the theoretical investigations of higher algebra.

The *common* system having 10 for a base is used for practical computations.

387. In the equation

$$b^x = N$$

supposing b to have any positive value other than 1,

when $N = 1$, the value of x is 0,

when $N = b$, the value of x is 1.

Hence

In any system the logarithm of 1 *is* 0, *and the logarithm of the base is* 1.

388. When N is equal to any integral power of the base, as $b^1, b^2, b^3, \ldots$ or $b^{-1}, b^{-2}, b^{-3}, \ldots$ the value of x is an integer. Hence

The logarithm of a number is integral when the number is equal to an integral power of the base.

When the number is not equal to an integral power of the base, the logarithm is expressed either exactly or approximately by fractions. See Art. 390.

389. By computing the values of x for any given series of values of N, say for all positive integers from 1 to 100, and arranging the results in a table, we should have a ***table of logarithms.***

390. NOTE ON THE COMPUTATION OF LOGARITHMS. Although a computer of logarithms would use short methods afforded by the higher branches of mathematics, it is well for the beginner to assure himself that the computation of a logarithm is quite within the reach of elementary methods. For the purpose of showing this, we shall here compute the common logarithm of 2 by solving the exponential equation

$$10^x = 2.$$

Seeing that 10^0 $(= 1)$ is less than 2, while 10^1 $(= 10)$ is greater than 2, we conclude that the value of x is greater than 0 and less than 1; in other words, x is a proper fraction. Accordingly we let $x = \frac{1}{x'}$, thus changing the form of the given equation to

$$10^{\frac{1}{x'}} = 2,$$

and seek the value of x'. By raising each member to the x' power, we have

$$2^{x'} = 10.$$

Seeing that 2^3 is less, while 2^4 is greater, than 10, we conclude that the value of x' is 3 plus a proper fraction, which we represent by $\frac{1}{x''}$; that is, $x' = 3 + \frac{1}{x''}$.

Substituting this value of x' in the last equation, observing that $2^{3+\frac{1}{x''}} = 2^3 \times 2^{\frac{1}{x''}}$, we have

$$2^3 \times 2^{\frac{1}{x''}} = 10$$

or, by reducing,

$$2^{\frac{1}{x''}} = \tfrac{5}{4}$$

from which we seek the value of x''. By raising each member to the x'' power, we have

$$(\tfrac{5}{4})^{x''} = 2.$$

Seeing that $(\frac{5}{4})^3$ is less, while $(\frac{5}{4})^4$ is greater, than 2, we conclude that the value of x'' is 3 plus a proper fraction, which we represent by $\frac{1}{x'''}$, that is, $x'' = 3 + \frac{1}{x'''}$.

Substituting this value of x'' in the last equation, and reducing, we have

$$(\tfrac{5}{4})^{\frac{1}{x'''}} = \tfrac{128}{125}$$

from which we seek the value of x'''. By raising each member to the x''' power, we have,

$$(\tfrac{128}{125})^{x'''} = \tfrac{5}{4}$$

or

$$(1.024)^{x'''} = 1.25$$

Seeing that $(1.024)^9$ is less,* while $(1.024)^{10}$ is greater, than 1.25, we conclude that the value of x''' is 9 plus a proper fraction.

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* The best way to find the 9th power of such a number as 1.024 is to use the Binomial Formula, making $a = 1$ and $b = 0.024$.

Substituting these values in the first four terms of $(a + b)^9$ gives

$$\begin{aligned} a^9 &= 1 \\ 9a^8b &= 0.216 \\ 36a^7b^2 &= 0.020736 \\ 84a^6b^3 &= \underline{0.001161\ \ldots} \\ & 1.237\ \ldots \end{aligned}$$

from which we see that $(1.024)^9$ is less than 1.25; for the rest of the

We might represent this fraction by $\frac{1}{x^{\text{iv}}}$ and continue the operation; but we have gone far enough to make the method clear.

Collecting the results, we find that

$$x=\frac{1}{x'}=\cfrac{1}{3+\cfrac{1}{x''}}=\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{x'''}}}=\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{x^{\text{iv}}}}}}$$

Neglecting the $\frac{1}{x^{\text{iv}}}$ we assume as an approximate value of x the complex fraction $\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{9}}}$ which reduced to a simple fraction is $\frac{28}{93}$, or the decimal 0.30107. The logarithm of 2 correct to five decimal places is 0.30103, from which we see that the error of our value of x is less than the half of one ten-thousandth.

Further steps would have made the approximation still closer.

As exercises, the equations $10^x=3$ and $10^x=5$ may be solved so far as to reach the approximate results

$$x=\cfrac{1}{2+\cfrac{1}{10+\cfrac{1}{2}}}=0.4772\ldots \qquad x=\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{9}}}}=0.6989\ldots$$

which differ but little from the values, correct to five places, of $\log 3=0.47712$, and $\log 5=0.69897$.

In this way we might compute all the logarithms given on the next page.

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terms are too small to change the first three figures of the result above obtained.

Substituting the same values in the first three terms of $(a+b)^{10}$ gives

$$\begin{aligned} a^{10}&=1\\ 10\,a^9\,b&=0.24\\ 45\,a^8\,b^2&=\underline{0.02592}\\ &1.26592\end{aligned}$$

from which we see that $(1.024)^{10}$ is greater than 1.25; for even the first three terms amount to 1.26 . . .

Common Logarithms of Numbers from 1 to 100.

Num.	Log.	Num.	Log.	Num.	Log.	Num.	Log.	Num.	Log.
1	0.0000	21	1.3222	41	1.6128	61	1.7853	81	1.9085
2	0.3010	22	1.3424	42	1.6232	62	1.7924	82	1.9138
3	0.4771	23	1.3617	43	1.6335	63	1.7993	83	1.9191
4	0.6021	24	1.3802	44	1.6435	64	1.8062	84	1.9243
5	0.6990	25	1.3979	45	1.6532	65	1.8129	85	1.9294
6	0.7782	26	1.4150	46	1.6628	66	1.8195	86	1.9345
7	0.8451	27	1.4314	47	1.6721	67	1.8261	87	1.9395
8	0.9031	28	1.4472	48	1.6812	68	1.8325	88	1.9445
9	0.9542	29	1.4624	49	1.6902	69	1.8388	89	1.9494
10	1.0000	30	1.4771	50	1.6990	70	1.8451	90	1.9542
11	1.0414	31	1.4914	51	1.7076	71	1.8513	91	1.9590
12	1.0792	32	1.5051	52	1.7160	72	1.8573	92	1.9638
13	1.1139	33	1.5185	53	1.7243	73	1.8633	93	1.9685
14	1.1461	34	1.5315	54	1.7324	74	1.8692	94	1.9731
15	1.1761	35	1.5441	55	1.7404	75	1.8751	95	1.9777
16	1.2041	36	1.5563	56	1.7482	76	1.8808	96	1.9823
17	1.2304	37	1.5682	57	1.7559	77	1.8865	97	1.9868
18	1.2553	38	1.5798	58	1.7634	78	1.8921	98	1.9912
19	1.2788	39	1.5911	59	1.7709	79	1.8976	99	1.9956
20	1.3010	40	1.6021	60	1.7782	80	1.9031	100	2.0000

Fundamental Propositions.

391. *To find the logarithm of a product.*

Let N and N' be any two numbers, and let x and x' be their logarithms to the base b.

Then

$$N = b^x$$
$$N' = b^{x'}.$$

Multiplying,

$$NN' = b^{x+x'}$$

whence it appears (Art. 384) that $x + x'$ is the logarithm of NN'.

The reasoning would be similar if there were three or more factors instead of two.

Hence

The logarithm of a product is equal to the sum of the logarithms of its factors.

392. *To find the logarithm of a quotient.*

Let N be the dividend and N' the divisor, and let x and x' be their logarithms as before.

Then $$N = b^x$$
$$N' = b^{x'}.$$

Dividing, $$N \div N' = b^{x-x'}$$
whence it appears (Art. 384) that $x - x'$ is the logarithm of $N \div N'$.

Hence

The logarithm of a quotient is equal to the logarithm of the dividend less the logarithm of the divisor.

393. *To find the logarithm of a power.*

Let $$N = b^x.$$

Taking the m^{th} power, $$N^m = b^{mx}$$
whence it appears (Art. 384) that mx is the logarithm of N^m.

Hence

The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

394. *To find the logarithm of a root.*

Let $$N = b^x.$$

Taking the n^{th} root, $$\sqrt[n]{N} = b^{\frac{x}{n}},$$

whence it appears (Art. 384) that $\frac{x}{n}$ is the logarithm of $\sqrt[n]{N}$.

Hence

The logarithm of a root of a number is equal to the logarithm of the number divided by the index of the root.

395. Briefly expressed in formulas the propositions just proved are as follows:

(1) $\log(N N' N'' \ldots) = \log N + \log N' + \log N'' + \ldots$

(2) $\log(N \div N') = \log N - \log N'.$

(3) $\log N^m = m \log N.$

(4) $\log \sqrt[n]{N} = \frac{1}{n} \log N.$

396. For the purpose of illustrating these formulas we shall select some simple examples coming within the range of the table on page 240. Thus we shall justify what was said in Art. 381 by showing how the multiplication or division of numbers can be performed by adding or subtracting their logarithms, and how the powers or the roots of numbers can be found by multiplying or dividing their logarithms.

I. Find by logarithms the product of 5 and 7.

Referring to the table, we find the logarithms of 5 and 7. These added give 1.5441 as the logarithm of 5 × 7. Referring again to the table, we find that 1.5441 is the logarithm of 35. Thus we have learned by the use of logarithms that the product of 5 and 7 is 35.

$$\begin{aligned} 0.6990 &= \log 5 \\ \underline{0.8451} &= \log 7 \\ \text{Sum, } 1.5441 &= \log(5 \times 7) \\ &= \log 35 \end{aligned}$$

Hence $5 \times 7 = 35$.

II. Find by logarithms the product of 3, 4, and 7.

By adding the logarithms of 3, 4, and 7, we find the logarithm of their product to be 1.9243, which stands in the table as the logarithm of 84. The product sought, then, is 84.

$$\begin{aligned} 0.4771 &= \log 3 \\ 0.6021 &= \log 4 \\ \underline{0.8451} &= \log 7 \\ \text{Sum, } 1.9243 &= \log(3 \times 4 \times 7) \\ &= \log 84 \end{aligned}$$

Hence $3 \times 4 \times 7 = 84$.

III. Find by logarithms the product of 2 and 3.

$$\begin{aligned} 0.3010 &= \log 2 \\ \underline{0.4771} &= \log 3 \\ \text{Sum, } 0.7781 &= \log(2 \times 3) \\ 0.7782 &= \log 6. \end{aligned}$$

Here the sum of the logarithms of 2 and 3 fails to agree with the logarithm of 6 as given in the table. But it is to be remembered that the values of the logarithms given in the table are only *approximate* values, being expressed in each case to the *nearest ten-thousandth*. The value actually expressed in the table may be greater or less than the true value by an amount not exceeding the half of a ten-thousandth. This half of a ten-thousandth is called *the possible error of a logarithm* in this table. The sum or difference of two logarithms would be affected by a possible error twice as great as that affecting a single logarithm, that is, by a possible error of *one* ten-thousandth. It is plain, therefore, that the logarithm of 2×3 agrees with the logarithm of 6 *within the limit of possible error.*

It follows, from what has just been said, that the sum of a large number of logarithms would be affected by a large *possible* error. This is true; and yet the *probable* error is small. For, since the actual errors of logarithms are sometimes positive and sometimes negative, the chances are, that most of the errors of one kind are offset by errors of the opposite kind, leaving the *probable* error of the result a small one.

IV. Divide 98 by 14, using logarithms.

By subtracting the logarithm of 14 from the logarithm of 98, we find the logarithm of the quotient to be 0.8451, which stands in the table opposite 7.

$$\begin{aligned} 1.9912 &= \log 98 \\ 1.1461 &= \log 14 \\ \hline 0.8451 &= \log(98 \div 14) \\ &= \log 7 \end{aligned}$$

Hence $98 \div 14 = 7$.

V. Find by logarithms the third power of 4.

Multiplying the logarithm of 4 by 3, we find the logarithm of 4^3 to be 1.8063, which agrees with the logarithm of 64 given in the table *within the limit of possible error.*

$$\begin{aligned} 0.6021 &= \log 4 \\ 1.8063 &= \log 4^3 \\ 1.8062 &= \log 64 \end{aligned}$$

Hence $4^3 = 64$.

VI. Find by logarithms the fourth root of 81.

Dividing the logarithm of 81 by 4, we find the logarithm of $\sqrt[4]{81}$ to be 0.4771, which agrees with the logarithm of 3 as given in the table.

$$\begin{aligned} 1.9085 &= \log 81 \\ 0.4771 &= \log \sqrt[4]{81} \\ &= \log 3 \end{aligned}$$

Hence $\sqrt[4]{81} = 3$.

VII. Find by logarithms the value of $32^{\frac{3}{5}}$.

By finding $\frac{3}{5}$ of the logarithm of 32, we have 0.9031 as the logarithm of $32^{\frac{3}{5}}$, and this is the logarithm of 8 as given in the table.

$$\begin{aligned} 1.5051 &= \log 32 \\ 4.5153 &= \log 32^3 \\ 0.9031 &= \log 32^{\frac{3}{5}} \\ &= \log 8 \end{aligned}$$

Hence $32^{\frac{3}{5}} = 8$.

VIII. Divide 16 by 48, using logarithms.

By subtracting the logarithm of 48 from the logarithm of 16, the negative remainder -0.4771 is found. Now $+0.4771$ is the logarithm of 3, as given in the table; then (Art. 336) -0.4771 must be the logarithm of the reciprocal of 3, that is, of $\frac{1}{3}$.

$$\begin{array}{r l}
1.2041 &= \log 16 \\
\underline{1.6812} &= \log 48 \\
-0.4771 &= \log(16 \div 48) \\
&= \log \tfrac{1}{3}
\end{array}$$

Hence $16 \div 48 = \frac{1}{3}$.

397. Exercises.

Using logarithms,

1. Multiply 15 by 6.
2. Multiply 31 by 3.
3. Multiply 18 by 4.
4. Multiply 2 by 5.
5. Multiply 4 by 10.
6. Multiply 8 by 10.
7. Multiply 10 by 10.
8. Divide 18 by 3.
9. Divide 48 by 6.
10. Divide 88 by 8.
11. Divide 92 by 4.
12. Divide 60 by 6.
13. Find the 2d power of 7.
14. Find the 5th power of 2.
15. Find the 3d power of 3.
16. Find the sq. root of 49.
17. Find the sq. root of 100.
18. Find the sq. root of 36.
19. Find the 3d root of 64.
20. Find the $\frac{4}{3}$ power of 27.
21. Find the $\frac{1}{2}$ power of 81.
22. Find the $\frac{5}{3}$ power of 8.
23. Divide 10 by 50.
24. Divide 20 by 80.
25. Divide 5 by 10.
26. Divide 4 by 100.
27. Divide 25 by 100.
28. Divide 1 by 5.
29. Multiply 30 by $\frac{1}{3}$.
30. Multiply 70 by $\frac{1}{10}$.

Exercises of this simple kind should be used until the general principles of logarithms become thoroughly familiar. Although the logarithms here used are *common* logarithms, it should be remembered that the same examples could have been worked with logarithms in any other system.

The Properties of Common Logarithms.

398. We have seen (Art. 388) that the logarithms of integral powers of the base are integers. In the common system, therefore, the integral logarithms are the following:

$$\begin{array}{lr|lr}
\log 1 & = 0 & \log 1 & = 0 \\
\log 10 & = 1 & \log 0.1 & = -1 \\
\log 100 & = 2 & \log 0.01 & = -2 \\
\log 1000 & = 3 & \log 0.001 & = -3 \\
\log 10000 & = 4 & \log 0.0001 & = -4 \\
\text{and so on} & & \text{and so on.} &
\end{array}$$

399. From this table we see that the logarithm of any number between

1 and 10 is $0 + a$ *fraction*
10 and 100 is $1 + a$ *fraction*
100 and 1000 is $2 + a$ *fraction*
1000 and 10000 is $3 + a$ *fraction*

and so on.

Also that the logarithm of any number between

1 and 0.1 is $-1 + a$ *fraction*
0.1 and 0.01 is $-2 + a$ *fraction*
0.01 and 0.001 is $-3 + a$ *fraction*

and so on.

400. The integral part of a logarithm is called its ***characteristic;*** and the fractional part, which is commonly expressed as a decimal, is called its ***mantissa.***

Thus, in the logarithm of 45, which is 1.6532, the characteristic is 1, and the mantissa is .6532; and in the logarithm of 2, which is 0.3010, the characteristic is 0, and the mantissa is .3010.

401. *If two numbers are expressed by the same series of significant figures, but in different positions with reference to the decimal point, their logarithms in the common system have the same mantissa, but different characteristics.*

Proof. It is plain that either of the numbers could be made equal to the other by changing the position of its decimal point, which would be the same as multiplying or dividing the number by some integral power of 10. The greater of the two numbers, then, is equal to the product of the smaller by some integral power of 10, and the *logarithm* of the greater number is equal to the *logarithm* of the smaller *plus* the logarithm of an integral power of 10. But the logarithm of an integral power of 10 is an integer. Hence the logarithm of the greater number exceeds the logarithm of the smaller by just an integer, which shows that the characteristics differ, but that the two mantissas, if they are both positive, have the same value. In the next article it is shown how logarithms are always written so as to have positive mantissas.

402. This reasoning is illustrated by the following examples, wherein, from the known logarithm of 45, are found the logarithms of various numbers expressed by the same significant figures, 45, in different positions with reference to the decimal point.

Thus,

to	$\log 45 = 1.6532$
adding	$\log 100 = 2$
we find	$\log 4500 = 3\ 6532$
(See Art. 391.)	
to	$\log 45 = 1.6532$
adding	$\log 1000 = 3$
we find	$\log 45000 = 4.6532$
from	$\log 45 = 1.6532$
subtracting (Art. 392)	$\log 10 = 1$
we find	$\log 4.5 = 0.6532$

We now select an example for the purpose of showing how the logarithm of a decimal fraction is written.

From	$\log 45 = 1.6532$
subtracting	$\log 1000 = 3$
we find	$\log 0.045 = \bar{2}.6532$

The simple negative value of this remainder is -1.3468, and this is the value of the logarithm of 0.045. But there is another form of expressing it. If the remainder is put in the form $-3 + 1.6532$, the integral part $-3 + 1$, equal to -2, is negative, and the fractional part .6532 is positive. These two parts are written together in the form $\bar{2}.6532$, the minus sign being written over the 2 to indicate that it alone is negative. By this form of expression we avoid the inconvenience of a negative mantissa.

Moreover, this form is needed to make the statement at the beginning of Art. 401 applicable to the logarithms of decimals as well as to the logarithms of integers. The number 0.045 being less than 0.1, but greater than 0.01, its logarithm is less than -1 and greater than -2; and so we have found it to be either -1 united with a negative decimal, that is, -1.3468, or -2 united with a positive decimal, that is, $\bar{2}.6532$. Of these two equivalent forms the latter is always chosen, and thus it becomes a rule that

The mantissa of a logarithm is always positive.

403. The examples given in the last article, with others directly derived from them, arranged in a series, will show how the position of the highest significant figure of a number may be used to determine the characteristic of its logarithm.

Thus,

$$\begin{array}{lrcl}
\cdot & \cdot & \cdot & \cdot \\
\log & 45000 & = & 4.6532 \\
\log & 4500 & = & 3.6532 \\
\log & 450 & = & 2.6532 \\
\log & 45 & = & 1.6532 \\
\log & 4.5 & = & 0.6532 \\
\log & 0.45 & = & \bar{1}.6532 \\
\log & 0.045 & = & \bar{2}.6532 \\
\log & 0.0045 & = & \bar{3}.6532 \\
\log & 0.00045 & = & \bar{4}.6532 \\
\cdot & \cdot & \cdot & \cdot
\end{array}$$

the series extending indefinitely both ways.

By comparing the position of the highest significant figure, 4, with the value of the characteristic in each case, we see, that

When the highest significant figure is

in the units' place,	the characteristic is		0
one place to the left,	"	"	+1
one place to the right,	"	"	−1
two places to the left,	"	"	+2
two places to the right,	"	"	−2
three places to the left,	"	"	+3
three places to the right,	"	"	−3

and so on.

Hence the practical rule for finding the characteristic:

Count from the units' place to the place of the highest significant figure of the number. The number obtained by so counting is the value of the characteristic; positive, if the counting was towards the left, and negative, if towards the right.

404. The characteristic, being readily found by the foregoing rule, need not be given in a table of logarithms. We only use the table for finding the mantissa corresponding to any given series of significant figures. The table on p. 240, therefore, contains more than is needed. By omitting the characteristics and decimal points, that table can be reduced to the following more compact form:

N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956

In this table the mantissa of the logarithm of a number expressed by any one or two figures is thus found:

Seek the first figure of the number in the left-hand *margin of the table, the second figure at the top of the table, and the required mantissa is found in the same line with the first figure, and in the same column with the second.*

405. Exercises.

Find the logarithm

31. Of 54.
32. Of 930000.
33. Of 3900.
34. Of 99000.
35. Of 6.9.
36. Of 0.78.
37. Of 5000.
38. Of 0.00079.
39. Of 0.27.
40. Of 0.00006.

Find the number corresponding

41. To 4.8129.
42. To 1.5185.
43. To 0.8261.
44. To 5.6335.
45. To 2.9031.
46. To $\bar{1}.2041$.
47. To $\bar{3}.6721$.
48. To $\bar{4}.9445$.
49. To $\bar{2}.7160$.
50. To $\bar{1}.0414$.

Tables of Logarithms.

406. The tables of logarithms given in the foregoing pages have served the purposes of illustration, but they are not large enough for practical use. Tables for practical use, however, are all constructed in the form of the table on p. 248; that is, the mantissas, printed without decimal points, fill the body of the table, and the figures of the corresponding numbers are found partly in the left-hand margin, and partly at the top of the table.

When mantissas are given to four places of decimals, the table is usually constructed so as to give the mantissa for every number expressed by three figures. The first two figures are found in the left-hand margin, and the third at the top of the table. Such a table, known as a ***four-place table of logarithms,*** is given on the two following pages.

FOUR-PLACE LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

A four-place table is useful for many kinds of work where only a moderate degree of accuracy is required. For work requiring higher degrees of accuracy, tables of five, six, seven, and even ten places, are used. A five-place table usually gives three figures in the left-hand margin and one more at the top; and a seven-place table, four in the left-hand margin and one more at the top.

But all tables are used in the same general way, both for finding the logarithms of given numbers and for finding the numbers corresponding to given logarithms. One who is familiar with the use of a smaller table will easily pass to larger tables.

For the purposes of elementary instruction, a four-place table may be chosen to save time; since experience has shown, that, in a long piece of work, the time consumed in using a four-place table is only one-half that consumed in using a five-place, and one-fourth that consumed in using a seven-place table.

407. Exercises.

Find the logarithm

51. Of 287.
52. Of 54.7
53. Of 9.99.
54. Of 3460000.
55. Of 0.0496.
56. Of 0.000987.

Find the number corresponding

57. To 3.8876.
58. To 0.9952.
59. To $\bar{2}.0253$.
60. To $\bar{1}.1673$.
61. To $\bar{4}.6201$.
62. To $\bar{3}.6010$.

Interpolation.

408. A table of logarithms is not limited in its use to the numbers found in its margin. The logarithm of any number, however long the series of its significant figures, can be found from any given table by a process called *interpolation.* The nature of this process may be learned from the following examples.

409. ILLUSTRATIVE EXAMPLES. I. Find the logarithm of 3254.

The table gives 5119 for the mantissa of the logarithm of 3250, and in the next column 5132 for the mantissa of the logarithm of 3260. The given number comes between the two numbers 3250 and 3260, separating the difference between them into two parts, 4 and 6. A mantissa coming between 5119 and 5132, and separating the difference in a similar way, that is, into parts proportional to 4 and 6, is taken as the mantissa for the given number. This mantissa is found by adding $\frac{4}{10}$ of 13 (=5.2) to the mantissa 5119, or by subtracting $\frac{6}{10}$ of 13 (= 7.8) from 5132. The result in either case, when expressed to the nearest unit in the fourth place, is 5124.

NUMBER.	MANTISSA.
3250	5119
4	5.2
3254	5124
6	7.8
3260	5132

To this mantissa we add the characteristic, and have

$$\log 3254 = 3.5124.$$

In the foregoing process it is taken for granted that the differences between numbers are proportional to the differences between their logarithms. Now, this is not true; yet it comes near the truth, *when*, as here, *the differences between the numbers are small in comparison with the numbers themselves.* In fact, a properly constructed table of logarithms permits us to assume as a principle, for the purpose of interpolation, that *small differences between numbers are proportional to the differences between their logarithms.* In the table we are now using, the difference never exceeds $\frac{1}{100}$ of either of the numbers that differ.

This principle may be stated in another form. The difference between 3250 and 3260 is 10, and is called *the tabular difference of the numbers.* The difference between the two consecutive mantissas 5119 and 5132 is 13, and is called *the tabular difference of the logarithms.* The given number 3254 is a *non-tabulated* number, and 5124 is a *non-tabulated mantissa.* Using these terms, the principle is thus stated:

The Principle of Proportional Parts.

A non-tabulated number and the mantissa of its logarithm fall within corresponding tabular differences, and separate them into proportional parts.

Interpolation based on this principle is known as ***simple interpolation,*** or ***interpolation by proportional parts.***

II. Find the logarithm of 253748.

NUMBER.	MANTISSA.
253000	4031
748	12.716
253748	4044
252	
254000	4048

Here the tabular difference of the numbers is 1000, and it is separated by the given number into two parts, 748 and 252. The tabular difference of the logarithms is 17, which is to be separated into two parts proportional to 748 and 252. It is enough to find one of these parts, $\frac{748}{1000}$ of 17, which is 12.716. Calling this 13, and adding it to 4031, we have 4044 for the mantissa sought. Adding the characteristic, we have

$$\log 253748 = 5.4044.$$

For the purpose of deriving a practical rule from the foregoing examples, it may be observed that the tabular difference of the numbers is always some integral power of 10. Hence the ratio of either part of this tabular difference to the whole of it is easily expressed as a decimal fraction. Considering that part which comes between the *smaller* tabular number and the given number, we see that its ratio to the whole tabular difference may be expressed by writing *the fourth and following figures of the given number with a decimal point before them.* Now, by the principle of proportional parts, we have merely to multiply the tabular difference of the logarithms by this ratio to find that part of the difference which comes between the *smaller* mantissa and the mantissa sought. When the part thus found is added to the smaller mantissa, the sum is the mantissa sought. By this process the smaller mantissa is said to be *corrected,* and what is added is called the *correction.*

The work for the two illustrative examples, therefore, may be written in the following brief form:

I. Find log 3254.

mant. for 325 = 5119
cor. 0.4 × 13 = 5
mant. for 3254 = 5124
log 3254 = 3.5124

II. Find log 253748.

mant. for 253 = 4031
cor. 0.748 × 17 = 13
mant. for 253748 = 4044
log 253748 = 5.4044

410. These examples suggest the following practical rule for finding the logarithm of a non-tabulated number:

Find the mantissa corresponding to the first three figures of the given number. Write the remaining figures with a decimal point before them. Multiply the decimal so expressed by the difference between the mantissa found and the next following mantissa in the table. Correct the mantissa first found by adding to it this product expressed to the nearest unit.

To the corrected mantissa prefix the proper characteristic.

Other Examples.

III. Find log 0.32987.

$$\begin{array}{lcr} \text{mant. for } 329 & = & 5172 \\ \text{cor. } 0.87 \times 13 & = & 11 \\ \text{mant. for } 32987 & = & \overline{5183} \\ \log 0.32987 & = & \bar{1}.5183 \end{array}$$

IV. Find log. 39.634.

$$\begin{array}{lcr} \text{mant. for } 396 & = & 5977 \\ \text{cor. } 0.34 \times 11 & = & 4 \\ \text{mant. for } 39634 & = & \overline{5981} \\ \log 39.634 & = & 1.5981 \end{array}$$

V. Find log 0.04985.

$$\begin{array}{lcr} \text{mant. for } 498 & = & 6972 \\ \text{cor. } 0.5 \times 9 & = & 5 \\ \text{mant. for } 4985 & = & \overline{6977} \\ \log 0.04985 & = & \bar{2}.6977 \end{array}$$

VI. Find log 9999.

$$\begin{array}{lcr} \text{mant. for } 999 & = & 9996 \\ \text{cor. } 0.9 \times 4 & = & 4 \\ \text{mant. for } 9999 & = & 1.\overline{0000} \\ \log 9999 & = & 4.0000 \end{array}$$

The last example is an extreme case. It is to be borne in mind that the logarithm of 9999 is *nearer* 4.000 than it is to 3.9999.

411. Exercises.

Find the logarithm

63. Of 48726.
64. Of 0.098324.
65. Of 7298800.
66. Of 9912834.
67. Of 0.001329.
68. Of 1.0024.
69. Of 2.9876.
70. Of 54.947.
71. Of 1.00099.
72. Of 1001.3.
73. Of 98.37.
74. Of 1.045.
75. Of 56.077.
76. Of 89.92.

412. The principle of proportional parts is also applied to find the number corresponding to a non-tabulated logarithm.

Illustrative Example. Find the number corresponding to 1.5848.

For the present, we disregard the characteristic and the place of the decimal point.

MANTISSA		NUMBER.	
5843		3840	
	5		4.2
5848		3844	
	7		5.8
5855		3850	

The given mantissa 5848 lies between 5843 and 5855, two successive mantissas found in the table, separating the difference between them into the parts 5 and 7. Corresponding to these two mantissas are the numbers 3840 and 3850, between which the number sought must be found. By the principle of proportional parts the number sought must separate the difference between 3840 and 3850 into two parts proportional to 5 and 7. Hence if $\frac{5}{12}$ of 10 be added to 3840, or if $\frac{7}{12}$ of 10 be subtracted from 3850, the result in either case will be the number sought, that is 3844, *so far as it can be expressed by four figures.*

The degree of accuracy belonging to a four-place table is not such as to warrant, in general, the finding of more than four figures of the number corresponding to a given logarithm.

The *figures* of the number sought are 3844. But the characteristic of the logarithm shows that the highest significant figure is *one* place to the left of the units' place. Hence the number sought, when expressed in the proper denomination, is 38.44.

413. This example suggests the following practical rule for finding the number corresponding to a non-tabulated logarithm:

Find in the table the mantissa next below the mantissa of the given logarithm, and write down the three figures of the corresponding number. Subtract this mantissa from the given mantissa. Divide ten times the remainder by the tabular difference of the two successive mantissas between which the given mantissa is found, expressing the *quotient to the nearest unit by one figure. Annex this figure to the three already written down. Thus are found four figures of the number sought.**

Place the decimal point as required by the given characteristic, annexing or prefixing zeros if any are needed.

The numerical work of the above example may be written in the following brief form:

$$\begin{array}{lrl} & \text{LOG.} & \text{NUM.} \\ \text{I.} & 1.5848 & 38.44 \\ & 43 & \\ & 12\overline{)50} & \\ & \overline{4} & \end{array}$$

Other Examples.

$$\begin{array}{lrlclrl} & \text{LOG.} & \text{NUM.} & & & \text{LOG.} & \text{NUM.} \\ \text{II.} & \bar{3}.4184 & 0.002621 & & \text{IV.} & 0.0415 & 1.1003 \\ & 83 & & & & 14 & \\ & 17\overline{)10} & & & & 39\overline{)100} & \\ & \overline{1} & & & & \overline{03} & \end{array}$$

$$\begin{array}{lrlclrl} & \text{LOG.} & \text{NUM.} & & & \text{LOG.} & \text{NUM.} \\ \text{III.} & 1.9999 & 99.98 & & \text{V.} & 3.0001 & 100.02 \\ & 96 & & & & 00 & \\ & 4\overline{)30} & & & & 43\overline{)100} & \\ & \overline{8} & & & & \overline{02} & \end{array}$$

414. Exercises.

Find the number corresponding

77. To 4.5507.
78. To $\bar{3}$.6801.
79. To 0.7321.
80. To $\bar{1}$.9490.
81. To 2.8509.
82. To $\bar{1}$.0060.
83. To 5.1084.
84. To $\bar{2}$.8437.
85. To $\bar{4}$.0009.
86. To 6.0066.
87. To 4.1932.
88. To 3.4900.
89. To $\bar{5}$.0122.
90. To 1.0171.
91. To 2.7783.
92. To 0.0420.
93. To 9.9162.
94. To $\bar{8}$.9060.

---

* In the early part of the table it is safe to find *five* figures. We should then divide 100 times the remainder by the tabular difference, expressing the quotient to the nearest unit by *two* figures. These would be annexed to the three already written down.

Computation by Logarithms.

415. Under this head it is only necessary to give the practical methods of computation, the principles upon which those methods are based having been already illustrated.

Multiplication.

416. ILLUSTRATIVE EXAMPLES. I. Find the continued product of 78.05, 0.6178, 34100, 10.009, and 0.0009.

We find the logarithm of each factor, add the logarithms, and find 4.1707 for the logarithm of the product. The product itself is 14814.

$$\begin{array}{rcl}
\log 78.05 & = & 1.8924 \\
\log 0.6178 & = & 9.7909-10 \\
\log 34100 & = & 4.5328 \\
\log 10.009 & = & 1.0004 \\
\log 0.0009 & = & \underline{6.9542-10} \\
 & & 24.1707-20 \\
\log \textit{product} & = & 4.1707 \\
\textit{product} & = & 14814.
\end{array}$$

To avoid the use of negative characteristics, which are inconvenient in adding, the logarithm of 0.6178 is written, not $\bar{1}.7909$, but $9.7909-10$; and the logarithm of 0.0009 is written, not $\bar{4}.9542$, but $6.9542-10$. The characteristic is increased by 10, and then the subtraction of 10 from the logarithm is expressed by writing -10 after it. When the sum of the logarithms has been found, the characteristic is reduced to its true value by subtracting the extra 10's thus introduced. In practice it is not usual to write -10 after the logarithm; but the fact that any characteristic has been increased by 10 is remembered, and allowed for at the end of the work, as in the following example.

II. Find the continued product of 3.26, 0.002578, 0.99, and 0.001177.

$$\begin{array}{rcl}
\log 3.26 & = & 0.5132 \\
\log 0.002578 & = & 7.4113 \\
\log 0.99 & = & 9.9956 \\
\log 0.001177 & = & \underline{7.0708} \\
 & & 24.9909 \\
\log \textit{product} & = & \bar{6}.9909 \\
\textit{product} & = & 0.000009793
\end{array}$$

417. Exercises.

95. Multiply 963472 by 884300.

96. Multiply the product of 0.0337 and 46.328 by 0.00934.

97. Find the continued product of 4.65, 97.432, 1067.8, and 0.09943.

98. Find the continued product of $4\frac{7}{9}$, $96\frac{5}{11}$, $33\frac{11}{15}$, $\frac{7}{12}$, and $\frac{1}{42}$.

Note. Common fractions may be reduced to decimals.

99. Find the circumference of a circle whose diameter is 43678.3 feet.

Note. $\pi = 3.1416$. See Note, p. 190.

100. Find the number of cubic inches in a cylinder 9 ft. 6 in. long and 1 ft. 8 in. in diameter.

Division.

418. The remainder obtained by subtracting a logarithm from 10 is called the ***arithmetical complement of the logarithm,*** or sometimes, for brevity, the ***cologarithm.***

The abbreviation ***colog*** is used for the words *the arithmetical complement of the logarithm of.*

Thus,

$$\log 2 = 0.3010$$
$$\text{colog}\ 2 = 9.6990.$$

The definition may be expressed by a formula thus,

$$10 - \log N = \text{colog}\ N.$$

419. We have seen that the logarithm of a quotient is found by subtracting the logarithm of the divisor from that of the dividend. This is expressed by the formula,

$$\log (M \div N) = \log M - \log N.$$

From the formula given in the preceding article we infer that

$$-\log N = \text{colog}\ N - 10.$$

Hence, by substitution in the formula just given,

$$\log(M \div N) = \log M + \operatorname{colog} N - 10.$$

That is,

The logarithm of a quotient is found by adding together the logarithm of the dividend and the cologarithm of the divisor, and subtracting 10 *from the result.*

420. Examples. I. Divide 63 by 7.

$$\begin{array}{rl}
\log 63 = & 1.7993 \\
\operatorname{colog} 7 = & 9.1549 \\
\hline
 & 10.9542 \\
\log \textit{quotient} = & 0.9542 \\
\textit{quotient} = & 9
\end{array}$$

II. Divide 0.01478 by 0.9243.

$$\begin{array}{rl}
\log 0.01478 = & 8.1697 - 10 \\
\operatorname{colog} 0.9243 = & 10.0342 \\
\hline
 & 18.2039 - 10 \\
\log \textit{quotient} = & \bar{2}.2039 \\
\textit{quotient} = & 0.015993
\end{array}$$

In examples like the above, it would be as easy to subtract the logarithm as to add the cologarithm; but when division by a number is only one step in a series of operations the use of the cologarithm is much more convenient. See Exs. III. and IV.

III. Find the value of $\dfrac{0.2735 \times 0.8432 \times 4598}{0.2988 \times 126.34}$.

$$\begin{array}{rl}
\log 0.2735 = & 9.4370 - 10 \\
\log 0.8432 = & 9.9259 - 10 \\
\log 4598 = & 3.6626 \\
\operatorname{colog} 0.2988 = & 10.5246 \\
\operatorname{colog} 126.34 = & 7.8984 \\
\hline
 & 41.4485 - 20 \\
\log \textit{value} = & 1.4485 \\
\textit{value} = & 28.09.
\end{array}$$

IV. Find the value of x from the proportion

$$0.3086 : 0.419 = 98.834 : x.$$

$$\begin{aligned}
\text{colog } 0.3086 &= 10.5106 \\
\log 0.419 &= 9.6222-10 \\
\log 98.834 &= \underline{1.9949} \\
& 22.1277-10 \\
\log x &= 2.1277 \\
x &= 134.19
\end{aligned}$$

To find the fourth term of a proportion: *Add together the cologarithm of the first term and the logarithms of the second and third. The sum diminished by* 10 *is the logarithm of the fourth term.*

V. Find the reciprocal of 387.45.

In other words, we are required to find the quotient of 1 divided by 387.45. Writing out the work in the usual form, we see that the logarithm of the reciprocal of a number is found by subtracting 10 from its cologarithm.

$$\begin{aligned}
\log 1 &= 0.0000 \\
\text{colog } 387.45 &= \underline{7.4118} \\
& 7.4118 \\
\text{subtract} & \underline{10} \\
\log \textit{reciprocal} &= \bar{3}.4118 \\
\textit{reciprocal} &= 0.002581
\end{aligned}$$

421. Exercises.

101. Divide 65.073 by 32.95.

102. Divide 0.004963 by 0.0007332.

103. Divide 100000 by 3.1416.

104. Find the value of $\dfrac{96.34 \times 352.67 \times 0.045}{0.039674 \times 1004 \times 993}$.

105. Find the value of $\dfrac{9\frac{3}{7} \times 46\frac{3}{11} \times 39\frac{1}{3} \times 49\frac{1}{7}}{10\frac{6}{7} \times 53\frac{2}{3} \times 92\frac{3}{5} \times 3\frac{9}{10}}$.

106. Find the value of $\dfrac{1.7322 \times 1.3641 \times 1.9325}{1.6288 \times 1.3196 \times 1.9264}$.

107. Divide the product 79.635×996.84 by the quotient $8.346 \div 0.0011223$.

108. Find the value of x from the proportion

$$2.499 : 3.983 = 5.446 : x.$$

109. Find the value of x from the proportion

$$0.0632 : 0.8414 = 0.0074 : x.$$

110. Find the value of x from the proportion

$$10000 : 370 = 1 : x.$$

111. Find the reciprocal of 34.984.
112. Find the reciprocal of 0.0025.
113. Find the reciprocal of 0.014567.
114. Find the value of $(32.125)^{-1}$.
115. Find the value of $(0.1875)^{-1}$.

Powers and Roots of Numbers.

422. EXAMPLES. I. Find the fifth power of 297.

$$\begin{array}{rr} \log 297 = & 2.4728 \\ \times & 5 \\ \hline \log (297)^5 = & 12.3640 \\ (297)^5 = & 2{,}312{,}000{,}000{,}000. \end{array}$$

The values found for the higher powers of numbers are affected by correspondingly greater degrees of uncertainty. The *possible error* in the logarithm of the given number, being multiplied by the exponent of the power, often makes the possible error in the logarithm of the power large enough to admit a variation in the fourth and even the third figure of the value found for the power itself. In the present case the possible error of the logarithm of the power is ± 2½ units in the last place of the mantissa, so that the first four significant figures of the power may be 2313 or 2311 as well as 2312.

II. Find the fourth power 0.9857.

$$\begin{array}{rr} \log 0.9857 = & 9.9937 - 10 \\ \times & 4 \\ \hline & 39.9748 - 40 \\ \log (0.9857)^4 = & \bar{1}.9748 \\ (0.9857)^4 = & 0.9436 \end{array}$$

III. Find the cube root of 63.

$$\begin{array}{rr} \log 63 = & 1.7993 \\ & \div 3 \\ \hline \log \sqrt[3]{63} = & 0.5998 \\ \sqrt[3]{63} = & 3.979 \end{array}$$

IV. Find the fifth root of 0.02814.

$$\begin{array}{rr} \log 0.02814 = & 48.4493 - 50 \\ & \div 5 \\ \hline & 9.6899 - 10 \\ \log \sqrt[5]{0.02814} = & \bar{1}.6899 \\ \sqrt[5]{0.02814} = & 0.4897 \end{array}$$

When the characteristic is negative, it should be increased before division, not by 10, but by as many 10's as there are units in the index of the required root, in this case by five 10's, or 50.

V. Find the fourth root of 0.005677.

Here four 10's are added before division.

$$\begin{array}{rr} \log 0.005677 = & 37.7541-40 \\ & \div 4 \quad\quad\quad\quad \\ \hline & 9.4385-10 \\ \log \sqrt[4]{0.005677} = & \bar{1}.4385 \quad\; \\ \sqrt[4]{0.005677} = & 0.2745 \quad\; \end{array}$$

VI. Find the $-\frac{2}{3}$ power of 0.041339.

$$\begin{array}{rr} \log 0.041339 = & 8.6164-10 \\ \times & 2 \\ \hline & 17.2328-20 \\ \log (0.041339)^2 = & 27.2328-30 \\ & \div 3 \quad\quad\quad\quad \\ \hline & 9.0776-10 \\ \log (0.041339)^{\frac{2}{3}} = & \bar{1}.0776 \quad\; \\ \text{colog } (0.041339)^{\frac{2}{3}} = & 10.9224 \quad\; \\ \log (0.041339)^{-\frac{2}{3}} = & 0.9224 \quad\; \\ (0.041339)^{-\frac{2}{3}} = & 8.364 \quad\; \end{array}$$

423. **Exercises.**

116. Find the sixth power of 4938.2.
117. Find the twentieth power of 17.
118. Find the third power of 0.025362.
119. Find the square root of 6.
120. Find the cube root of 7.
121. Find the square root of 0.009.
122. Find the cube root of 0.8.
123. Find the fourth root of 4.444.
124. Find the 48th root of 1.476.
125. Find the 30th root of 0.095.
126. Find the $\frac{2}{5}$ power of 256.75.
127. Find the $\frac{4}{3}$ power of 0.0561.
128. Find the $-\frac{3}{4}$ power of 965.
129. Find the value of $\sqrt{43.99 \times 0.0005693}$.
130. Find the value of $(39.6 \times 93.21)^{\frac{3}{4}}$.
131. Find the value of $(546.2 \times 89.344)^{-\frac{3}{7}}$.

132. Find the value of $\dfrac{1}{(36.44)^{\frac{1}{3}}}$.

133. Find the value of $\sqrt[15]{\dfrac{45963}{11234}}$.

134. Find the value of $(1.0025)^{10}$.

135. Find the value of $(1.01)^{100}$.

424. Miscellaneous Exercises.

Find by logarithms the value

136. Of $\dfrac{\sqrt[3]{43.6} \times \sqrt[4]{944.8}}{\sqrt{596.04} \times 48.922}$.

137. Of $\dfrac{(493)^4 \times (9.018)^5 \times (549.54)^3}{(513)^5 \times (8.447)^4 \times (6900)^2}$.

138. Of $\left(\dfrac{(63.92)^5 \times (0.0942)^4}{(54.48)^6 \times (0.953)^5}\right)^3$.

139. Of $\left(\dfrac{(5.593)^{\frac{1}{3}} \times (6.998)^3}{(7.004)^{\frac{1}{4}} \times (5.998)^4}\right)^5$.

140. Of $\sqrt[4]{\dfrac{(639)^3 \times (411.96)^4 \times (8.9)^5}{(744)^5 \times (411)^4 \times (0.96)^3}}$.

141. Of $\left(\dfrac{43.2 \times \sqrt[3]{8.476} \times \sqrt{0.0963}}{3964.53 \times (0.9986)^3}\right)^{\frac{2}{5}}$.

142. Of $\left(\dfrac{(0.328)^4 \times \sqrt{19.63} \times (0.075)^{-1}}{(9.28)^{\frac{1}{3}} \times (\frac{1}{4})^{\frac{1}{4}} \times (\frac{9}{17})^{\frac{7}{4}}}\right)^{\frac{2}{3}}$.

143. Of $\sqrt{\dfrac{49.326}{53.629}} \times \sqrt{\dfrac{68.433}{67.966}}$.

144. Of $\sqrt{\dfrac{3.764}{4.665}} + \sqrt{\dfrac{5.8824}{9.6225}}$.

145. Of $\sqrt[3]{\dfrac{0.0944}{0.1449}} \div \sqrt[3]{\dfrac{0.1432}{0.0899}}$.

146. Of $\sqrt[3]{\dfrac{0.049}{0.054}} - \sqrt[3]{\dfrac{0.388}{0.698}}$.

147. Of $\sqrt[4]{\dfrac{17.28}{19.41}} \times \sqrt[5]{\dfrac{24.32}{96.21}} \times \sqrt[6]{\dfrac{13.34}{6.95}}$.

With the given values $a = 2.432$, $b = 5.937$, $c = 7.098$, $x = 1.443$, $y = 1.586$, $z = 1.755$, compute the values of the following expressions:

148. $ax + by + cz.$

149. $\frac{ax}{b} + \frac{by}{c} + \frac{cz}{a}.$

150. $\frac{ax}{by} + \frac{by}{cz} + \frac{cz}{ax}.$

151. $\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right).$

152. $\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right).$

153. $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$

154. $\frac{abz}{cyx} + \frac{bcx}{azy} + \frac{cay}{bxz}.$

155. $\sqrt{ax^2 + by^2 + cz^2}.$

156. $\sqrt[3]{\left(\frac{a}{x}\right)^3 + \left(\frac{b}{y}\right)^3 + \left(\frac{c}{z}\right)^3}.$

157. $\sqrt{(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)}.$

158. $\frac{(a - x)^2 (b - y)^2 (c - z)^2}{abcxyz}.$

159. $\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)\left(1 - \frac{z}{c}\right).$

160. $(a + x)(a - x) + (b + y)(b - y) + (c + z)(c - z).$

161. $\sqrt{a^2 - x^2} + \sqrt{b^2 - y^2} + \sqrt{c^2 - z^2}.$

NOTE. Observe that $a^2 - x^2 = (a + x)(a - x).$

162. $\frac{a^2 - x^2}{b^2 - y^2} + \frac{b^2 - y^2}{c^2 - z^2} + \frac{c^2 - z^2}{a^2 - x^2}.$

163. $\frac{a^2 - x^2}{a^2} + \frac{b^2 - y^2}{b^2} + \frac{c^2 - z^2}{c^2}.$

Exponential Equations.

425. An exponential equation is easily solved by the use of logarithms.

Illustrative Example. Find the value of x in the equation $17^x = 47$.

Taking the logarithm of each member, $x \log 17 = \log 47$

Dividing each member by log 17, $x = \frac{\log 47}{\log 17} = \frac{1.6721}{1.2304}$

Again taking the logarithm of each member,

$$\log x = \log \tfrac{1.6721}{1.2304} = 0.2233 - 0.0900 = 0.1333$$

$$x = 1.359$$

426. Exercises.

164. Given $5^x = 100$, to find x.
165. Given $175^x = 1000$, to find x.
166. Given $3^x = 2$, to find x.
167. Given $25^x = 10000$, to find x.
168. Given $(1.002)^x = 1.5$, to find x.
169. Given $(1.04)^n = 3$, to find n.
170. Given $(1.045)^n = 5$, to find n.
171. Given $(1.06)^n = 2$, to find n.

Compound Interest by Logarithms.

427. Illustrative Examples. I. What is the amount of \$3784 at compound interest for forty years at 5 per cent per annum?

By substituting in the formula for compound interest, Art. 217, we have

$$a = 3784(1.05)^{40}.$$

Applying logarithms,

$$\log 1.05 = 0.0212$$
$$\log (1.05)^{40} = 0.8480$$
$$\log 3784 = 3.5779$$
$$\log a = 4.4259$$
$$a = 26660$$

Ans. About \$26660.

NOTE. The *possible* error of the logarithm of a is $20\frac{1}{2}$ ten-thousandths. Hence the value of a may be as small as 26540, or as great as 26790. This shows that we are certain only of the first two figures of the value of a, the third figure being open to some doubt. Thus we see the need of larger tables if we seek practically accurate and useful results in such examples as this.

II. What principal put at compound interest at $4\frac{1}{2}$ per cent per annum will amount to $10000 in twenty years?

By substituting in the formula,

$$10000 = p(1.045)^{20}$$

$$p = \frac{10000}{(1.045)^{20}}.$$

Applying logarithms,

$$\begin{aligned} \log 1.045 &= 0.0191 \\ \log (1.045)^{20} &= 0.3820 \\ \text{colog } (1.045)^{20} &= 9.6180 \\ \log 10000 &= 4.0000 \\ \log p &= 3.6180 \\ p &= 4150 \end{aligned}$$

Ans. About $4150.

The value of p in this example is called the *present worth* of $10000 payable twenty years hence.

III. At what rate per annum, compound interest, will $3600 amount to $5524.60 in eight years?

By substituting in the formula,

$$5524.60 = 3600(1+r)^8$$

$$1 + r = \sqrt[8]{\frac{5524.60}{3600}}.$$

Applying logarithms,

$$\begin{aligned} \log 5524.60 &= 3.7423 \\ \text{colog } 3600 &= \underline{6.4437} \\ &\ 8\overline{)0.1860} \\ \log 1 + r &= 0.0232 \\ 1 + r &= 1.055 \\ r &= 0.055 \end{aligned}$$

Ans. $5\frac{1}{2}$ per cent.

IV. In how many years will \$234 amount to \$420.35 at 5 per cent compound interest?

By substituting in the formula,

$$
\begin{aligned}
420.35 &= 234(1.05)^n\\
(1.05)^n &= \tfrac{420.35}{234}\\
\log(1.05)^n &= \log\tfrac{420.35}{234}\\
n\log 1.05 &= \log 420.35 - \log 234\\
n &= \frac{\log 420.35 - \log 234}{\log 1.05}.
\end{aligned}
$$

Applying logarithms,

$$
\begin{aligned}
\log 420.35 &= 2.6236\\
\log 234 &= \underline{2.3692}\\
& 0.2544\\
\log 1.05 &= 0.0212\\
n &= \tfrac{0.2544}{0.0212}\\
\log 0.2544 &= 9.4055\\
\operatorname{colog} 0.0212 &= \underline{11.6737}\\
\log n &= 1.0792\\
n &= 12.00
\end{aligned}
$$

Ans. In 12 years.

V. In how many years will a sum of money at compound interest double itself at 6 per cent per annum?

The condition is that $a = 2p$. Substituting $2p$ for a in the formula, we have

$$
\begin{aligned}
2p &= p(1.06)^n\\
(1.06)^n &= 2\\
n\log 1.06 &= \log 2\\
n &= \frac{\log 2}{\log 1.06} = \tfrac{0.3010}{0.0258}\\
\log n &= 9.4786 - 8.4031\\
\log n &= 1.0755\\
n &= 11.9
\end{aligned}
$$

Ans. In 11.9 years.

NOTE. The formula for compound interest

$$a = p(1 + r)^n$$

is based on the supposition that the interest is compounded *annu-*

ally. The rate *per annum* is represented by r. If the interest is compounded *semi-annually*, the formula must be changed by putting $\frac{r}{2}$ for r, and $2n$ for n, thus:

$$a = p\left(1 + \frac{r}{2}\right)^{2n}.$$

In the same way, if the interest is compounded *quarterly*, the formula becomes

$$a = p\left(1 + \frac{r}{4}\right)^{4n}.$$

And in general, supposing the interest to be compounded q times a year, the formula becomes

$$a = p\left(1 + \frac{r}{q}\right)^{qn}.$$

428. **Exercises.**

172. Find the amount of $387.50 for 11 years at 6 per cent, compound interest.

173. Find the amount of $1496 for 4 years at 7 per cent, compound interest.

174. If interest were compounded monthly at the rate of 6 per cent a year, what would be the amount of $100 at the end of one year?

175. How much more will $1000 earn in three years, if the interest is compounded quarterly, than if it is compounded annually, the rate per annum being 6 per cent?

176. What sum will amount to $5000 in ten years, compound interest, at 5 per cent per annum?

177. What sum will amount to $25000 in twenty years, compound interest, at 3 per cent per annum?

178. In twelve years $2170 at compound interest amounted to $4125.40. What was the rate?

179. At what rate per annum, compound interest. would $42000 amount to $60000 in five years?

180. At what rate per annum, compound interest, would a sum of money double itself in ten years?

181. At what rate per quarter, when interest is com-

pounded quarterly, would $1000 amount to $1500 in five years?

182. How long will it take $100 to earn $100 at 6 per cent per annum, compound interest?

183. How long will it take a sum of money to triple itself at 4 per cent per annum, compound interest?

184. How long will it take $55000 to earn $12000 at $5\frac{1}{2}$ per cent per annum, compound interest?

185. Interest being compounded semi-annually at the rate of $4\frac{1}{2}$ per cent per annum, how long will it take $7000 to earn $3500?

186. In the year 1870 the population of the United States was 39000000; in 1880 it was 50000000. What was the yearly rate of increase?

NOTE. The formula for compound interest is applicable to questions of this kind.

187. Supposing the yearly rate of increase in the population of the United States to continue the same from 1880 to 1890 that it was from 1870 to 1880, what will be the population in 1890? On a similar supposition, what will the population be in 1900?

188. Supposing the same rate of increase to continue, in how many years from 1880 will the population of the United States be doubled?

189. Apply logarithms to the solution of Ex. 55, p. 230.

Sinking Funds.

429. PROBLEM. A person deposits p dollars annually where the money accumulates at compound interest. What will the whole amount deposited and accumulated be at the end of n years from the time of making the first deposit, the rate of interest being r per annum?

SOLUTION. At the end of n years the first deposit will have been at compound interest n years, the second $n-1$ years, the third $n-2$ years, and so on. The last deposit, made at the beginning of the n^{th} year, will have been at interest 1 year.

Hence, Amount of first deposit $= p(1+r)^n$

Amount of second deposit $= p(1+r)^{n-1}$

. . . . . .

Amount of the last deposit but one $= p(1+r)^2$

Amount of the last deposit $= p(1+r)$.

The sum of these amounts is

$$p(1+r)+p(1+r)^2+\ .\ .\ .\ +p(1+r)^n.$$

This is a geometrical progression, of which the first term is $p(1+r)$, the last term $p(1+r)^n$, the number of terms n, and the common ratio $1+r$.

By substituting in the formula (Art. 373) $p(1+r)$ for a, and $1+r$ for r, the sum of all the terms is found to be

$$\frac{p(1+r)\left[(1+r)^n-1\right]}{(1+r)-1},\text{ or }\frac{p}{r}\left[(1+r)^{n+1}-(1+r)\right].$$

Representing by A the whole amount deposited and accumulated, the formula is

$$A=\frac{p}{r}\left[(1+r)^{n+1}-(1+r)\right].$$

APPLICATION. A person makes an annual deposit of \$600. What is the whole amount deposited and accumulated at the end of twenty years, interest being compounded annually at 4 per cent?

$$A=\frac{600}{0.04}\left[(1.04)^{21}-1.04\right].$$

Applying logarithms,

$$\begin{aligned}
\log 1.04 &= 0.0170\\
\log (1.04)^{21} &= 0.3570\\
(1.04)^{21} &= 2.275\\
(1.04)^{21}-1.04 &= 1.235\\
\hline
\log 1.235 &= 0.0917\\
\log 600 &= 2.7782\\
\operatorname{colog} 0.04 &= 11.3979\\
\log A &= 4.2678\\
A &= 18526.
\end{aligned}$$

Ans. \$18526.

430. If it be required to find what sum, p, deposited annually, will amount to A dollars in n years at a given rate, r, compound interest, the formula may be reduced to the form

$$p = \frac{Ar}{(1+r)^{n+1} - (1+r)}.$$

APPLICATION. What sum deposited annually will amount to \$10000 in twenty years, interest being compounded annually at 5 per cent?

$$p = \frac{10000 \times 0.05}{(1.05)^{21} - 1.05}$$

$$\begin{array}{rl}
\log(1.05) = & 0.0212 \\
\log(1.05)^{21} = & 0.4452 \\
(1.05)^{21} = & 2.787 \\
(1.05)^{21} - 1.05 = & 1.737 \\
\hline
\text{colog } 1.737 = & 9.7601 \\
\log 10000 = & 4.0000 \\
\log 0.05 = & 8.6990 \\
\log p = & 2.4591 \\
p = & 287.80
\end{array}$$

Ans. \$287.80.

NOTE. If the deposits are made *semi-annually*, *quarterly*, or q times a year, the above formulas are easily changed to suit the case by putting $\frac{r}{q}$ for r, and qn for n.

431. Exercises.

190. A clerk deposits annually \$175 in a savings bank which pays 4 per cent per annum, compound interest. What amount will he have accumulated in six years?

191. What will a saving of \$400 a year amount to in ten years, if the money is deposited where it draws 4½ per cent compound interest?

192. A young man saves 20 cents a day, and at the end of each year deposits the amount saved in a bank which pays 4 per cent, compound interest. If he began at the age of twenty, what will his savings amount to when he reaches the age of forty years?

NOTE. Reckon each year as 365 days.

193. What sum must be deposited annually with a trust company which allows 5 per cent, compound interest, to cancel at maturity a debt of $100000 maturing twenty years hence?

194. A town borrows $250000 for the purpose of constructing water-works, issuing for this amount bonds which will mature in 25 years. What sum must be annually set aside as a sinking fund to cancel the debt at maturity, supposing the sinking fund can be made to earn 4½ per cent per annum?

Annuities.

432. An annuity is a sum of money payable yearly to a person for a term of years or for his lifetime.

Problem. What sum of money must a person deposit now in an annuity office that he may henceforth draw from the office a dollars annually for n years, thus exhausting his deposit and the accumulated interest at the end of that time?

Solution. The principle adopted for the solution of questions of this kind is this, that *the original deposit, with its accumulation at a given rate of compound interest, shall be equal to the sum of all the yearly payments and their several accumulations up to the time of the last payment.*

Let P represent the original deposit, and r the rate of interest. The amount of P at the end of n years at compound interest is

$$P(1+r)^n.$$

This is the amount placed to the depositor's credit.

Against this is charged the amount of each payment from the time it was made to the end of the n years; that is,

Amount of first payment for $n-1$ years $= a(1+r)^{n-1}$

Amount of second payment for $n-2$ years $= a(1+r)^{n-2}$

. . . .

Amount of last payment but one $= a(1+r)$

Amount of last payment $= a$

The sum of all these amounts is

$$a + a(1+r) + \dots + a(1+r)^{n-1}.$$

This is a geometrical progression the sum of whose terms (Art. 373) is

$$\frac{a[(1+r)^n-1]}{(1+r)-1}, \text{ or } \frac{a}{r}[(1+r)^n-1]$$

which is the gross amount placed to the depositor's debit.

Since the account must balance,

$$P(1+r)^n=\frac{a}{r}[(1+r)^n-1]$$

whence

$$P=\frac{a[(1+r)^n-1]}{r(1+r)^n}.$$

APPLICATION. What is the cost of an annuity of \$1200 a year to run for thirty years from the present time, the rate of interest being 3 per cent?

$$P=\frac{1200\,[(1.03)^{30}-1]}{0.03\times(1.03)^{30}}.$$

Applying logarithms,

$$\begin{array}{rl}
\log 1.03 = & 0.0128 \\
\log (1.03)^{30} = & 0.3840 \\
(1.03)^{30} = & 2.421 \\
(1.03)^{30}-1 = & 1.421 \\
\hline
\log 1.421 = & 0.1526 \\
\log 1200 = & 3.0792 \\
\text{colog } 0.03 = & 11.5229 \\
\text{colog } (1.03)^{30} = & 9.6160 \\
\log P = & 4.3707 \\
P = & 23480
\end{array}$$

Ans. \$23480.

433. It may easily be shown that the above formula can be reduced to the form

$$a=\frac{Pr(1+r)^n}{(1+r)^n-1}$$

for the purpose of finding what annuity, *a*, for a given number of years would be afforded by a given deposit at a given rate of interest, and to the form

$$n = \frac{\log a - \log(a - Pr)}{\log(1 + r)}$$

for the purpose of finding how long a given deposit will afford a given annuity at a given rate of interest.

APPLICATION. How many years will an annuity of \$800 run against a deposit of \$8000 for which $4\frac{1}{2}$ per cent compound interest is allowed?

By the last formula above,

$$n = \frac{\log 800 - \log 440}{\log 1.045}.$$

Applying logarithms,

$$\begin{aligned} \log 800 &= 2.9031 \\ \log 440 &= 2.6435 \\ \log 800 - \log 440 &= 0.2596 \\ \log 1.045 &= 0.0191 \\ n &= \tfrac{0.2596}{0.0191} \\ \log 0.2596 &= 9.4143 \\ \text{colog } 0.0191 &= 11.7190 \\ \log n &= 1.1333 \\ n &= 13.59 \end{aligned}$$

Ans. Between 13 and 14 years.

NOTE. If the payments are made semi-annually, quarterly, or q times a year, the above formulas are easily changed to suit the case, by putting $\frac{r}{q}$ for r, and qn for n.

434. Exercises.

195. What is the cost of an annuity of \$1000 for 23 years at 5 per cent, compound interest?

196. What is the present worth of an annual pension of \$600 for 20 years at $4\frac{1}{2}$ per cent, compound interest?

197. Which is the more valuable, and how much, a pension of \$500 a year for twenty years, or an immediate

payment of $6000, compound interest being reckoned at 5 per cent per annum?

198. How large an annuity can be purchased with $30000, if it is to run thirty years at 3½ per cent, compound interest?

199. A man dying leaves his whole estate, valued at $250000, to a college, providing, however, that the college shall pay to his widow an annuity of $2000 during the rest of her life. Assuming 27 years as the length of time a person of her age is expected to live, and 4 per cent as the rate of compound interest, what is the present value to the college of the legacy?

200. How long would a deposit of $50000 afford an annuity of $3000, reckoning compound interest at 5 per cent?

201. A debt of $80000, bearing compound interest at 5 per cent per annum, is to be paid off in twenty annual payments, the first payment to be made one year hence. What must each annual payment be?

202. In a certain country having now a population of 6000000, the natural rate of increase in population is $\frac{1}{5}$ of one per cent per annum; but in consequence of emigration the population decreases. If emigration should continue at the rate of 20000 people every year, how many years before this country would be wholly devoid of inhabitants?

SECTION XIV.

MISCELLANEOUS PROBLEMS AND EXERCISES.

1. What sum of money is that of which a fourth part and a fifth part together amount to $7740?

2. A and B started in trade at the same time, A with $2750, and B with $3000. A lost half as much again as B, and B then had left half as much again as A. How much did each lose?

3. Divide 21000 acres of land among four persons so that A's share shall be $\frac{2}{3}$ of B's, B's share $\frac{4}{5}$ of C's, and C's share $\frac{6}{7}$ of D's.

4. A and B can do a piece of work in $11\frac{1}{5}$ days, but A can do it alone in 21 days. How long would B take to do it alone?

5. One arm of a lever is 7 feet and the other 8 feet long, and the two loads balancing one another on these arms differ by 50 pounds. How many pounds does each load weigh?

6. At what time between 2 and 3 o'clock do the hour and minute hands point in exactly opposite directions?

7. At a certain time between 9 and 10 o'clock, the minute-hand is $15\frac{2}{3}$ minute-spaces behind the hour-hand. What o'clock is it?

8. Two boys had a mile race with their bicycles. Their rates were as 4 to 5. The loser, although he had a start of half a minute, was beaten by 176 yards. What was the rate of each?

9. A merchant bought a piece of cloth at $3.75 a yard, and, intending to gain 20 %, sold it at $4.50 a yard; but, the piece proving to be $2\frac{1}{2}$ yards shorter than he supposed it

to be when he paid for it, he really gained only $14\frac{3}{13}$ %. How many yards did he suppose were in the piece?

10. A boy being asked to divide one half of a certain number by 4, and the other half by 6, and to add the quotients, tried to get the answer at one step by dividing the whole number by 5; but his answer was too small by 2. What was the number?

11. Two steamboats make the same trip, one in $7\frac{1}{2}$ hours, and the other in 8 hours. During the trip the faster boat overtakes the other and passes her, giving the passengers an opportunity to observe the speed of the two. From the moment when the prow of the faster boat is abreast the stern of the slower to the moment when the stern of the faster is abreast the prow of the slower is observed to be just 6 minutes. Each boat is 300 feet long. Find the speed of each boat, and the length of the trip in miles.

12. A father dying leaves children, who by his will are to share his property as follows: the first takes $1000 and one-tenth of what remains; then the second takes $2000 and one-tenth of what then remains; then the third takes $3000 and one-tenth of what then remains; and so on, each taking $1000 more than the one before him and one-tenth of the remainder. But it turns out that the property is thus equally divided among all the children. What is the amount of property divided, the number of children, and the share of each?

13. What number is that of which the m^{th} part and the n^{th} part together amount to a?

14. When a pounds of a commodity cost m dollars, what is the cost of b pounds?

15. What is the interest on k dollars for l years, at c per cent per annum?

16. George can run as far in a minutes as Albert can in b minutes. How far can George run while Albert runs m rods?

17. A messenger who travels a miles in k hours goes from

one city to another in n hours. In how many hours can another go who travels a' miles in k' hours?

18. One laborer earns as much in a days as another does in b days. The first earns k dollars in l days. How much does the other earn in the same time?

19. What number subtracted from both numerator and denominator of the fraction $\frac{a}{b}$ will change its value to $\frac{m}{n}$?

20. A merchant sells k yards of cloth for as much money as l yards cost him. What per cent does he gain?

21. One merchant owes another $2500 payable after 3 months, $3200 payable after 8 months, and $1300 payable after 9 months. When may the whole be paid without loss of interest to either party?

22. Generalize the foregoing by finding when a whole sum $a+b+c+d$ may be paid, if a is payable after k months, b after l months, c after m months, and d after n months. What rule of arithmetic is expressed by the result?

23. A straight line A B is divided at the point M, so that the part A M is to the part M B as 13 to 17; but, if the point M be moved 4 inches towards B, the part A M will then be to the part M B as 15 to 19. What is the length of the whole line?

24. What is the price of corn when 5 bushels of corn and $8.75 in money are given for 10 days' labor, and, at the same rates, 9 bushels of corn and $13.25 in money for 16 days' labor?

25. A and B can do a piece of work together in 8 days. A works alone 4 days, and then both finish it in 5 days more. In what time could each have done it alone?

26. I have a certain sum of money at interest at a certain rate. My brother A, having $1000 more than I, at interest, at a rate 1 % higher, receives $80 more interest than I do. My brother B, having $1500 more than I, at interest 2 % higher, receives $150 more than I do. How much money have I at interest, and at what rate?

27. A cast-iron flask full of mercury weighs 62 kilograms; but under water it weighs only 56.36 kilograms. How much mercury is there in the flask, supposing the specific gravity of the cast-iron to be 7.1, and that of mercury 13.6?

28. The sum of two fractions is $\frac{a}{b}$, their difference $\frac{c}{d}$; what are the fractions?

29. Two numbers are as $1\frac{1}{2}$ to $2\frac{2}{3}$; but, if each be increased by 15, they will then be as $1\frac{2}{3}$ to $2\frac{1}{2}$.

30. The area of a triangle varies as the base and height jointly. The bases of two triangles are as 8 to 11, and their heights as 7 to 13. What is the ratio of their areas?

31. If $a : b = 3 : 2$, what is the value of $a + b : a - b$?

32. If $a : b = 5 : 3$, what is the value of $\frac{a}{b} + \frac{b}{a}$?

If $a : b = 4 : 5$, what is the value

33. Of $1 - \frac{a^2}{b^2}$? 34. Of $\sqrt{\frac{b^2 - a^2}{a^2}}$? 35. Of $\sqrt{\frac{b^2}{b^2 - a^2}}$?

36. Separate the number a into three parts, x, y, and z, such that $x : y = m : n$, and $y : z = p : q$.

37. Separate the number a into four parts, w, x, y, and z, such that $w : x = m : n$, $x : y = p : q$, and $y : z = r : s$.

38. A traveller having spent 864 pounds sterling in England, 5288 francs in France, and 4264 marks in Germany, finds that the whole amount spent is equal to \$6234. He reckons 4 pounds sterling to be worth 21 cents more than 100 francs, and 25 francs to be worth $6\frac{1}{4}$ cents more than 20 marks. At what values respectively does he reckon the pound, the franc, and the mark?

39. Find three numbers which will satisfy the following conditions: if 6 be added to the first and to the second, the sums will be to each other as 2 to 3; if 5 be added to the first and to the third, the sums will be to each other as 7 to 11; and, if 36 be subtracted from the second and from the third, the remainders will be to each other as 6 to 7.

40. In an experiment, 3 lbs. of maple wood, 1 lb. of cork, and 5 lbs. of lead, being tied together and weighed under water, are found to lose 8.40 lbs. In another, 2 lbs. of maple wood, 1 lb. of cork, and 6 lbs. of lead, lose 7.23 lbs. In a third, 1 lb. of maple wood, 2 lbs. of cork, and 8 lbs. of lead, lose 10.31 lb. From these results compute the specific gravity of maple wood, of cork, and of lead.

41. Prove that the difference of the squares of two consecutive integers is equal to the sum of those integers.

42. Show that the sum of the cubes of any three consecutive integers is divisible by the second of them.

43. If the difference of two fractions is $\frac{m}{n}$, show that m times their sum is equal to n times the difference of their squares.

44. There are three numbers such that the product of each pair, when divided by the other number, gives the quotients 3, 12, and 27 respectively. Find the numbers.

Generalize this question, representing the three quotients by a, b, and c.

45. Find the number whose cube root is one-fifth of its square root.

46. A person who can walk forwards four times as fast as he can walk backwards undertakes to walk a certain distance, one-fourth of it backwards, in a stated time. He begins by walking backwards; but, finding his speed one-fifth of a mile an hour less than he expected, he is obliged to walk forwards two miles an hour faster than he intended in order to finish his task within the stated time. How fast did he expect to walk forwards, and how fast backwards?

47. Two men started together on a journey of 210 miles. One travelled a mile an hour faster than the other, and arrived at his journey's end five hours sooner. How many miles an hour did each travel?

48. The fore wheel of a carriage makes six more turns than the hind wheel in going 120 yards, and the circumfer-

ences differ by one yard. Find the circumference of each wheel.

49. Dividing 800 by a certain divisor gives a quotient and no remainder. If the divisor should be diminished by 20, the quotient would be greater by 2, and still there would be no remainder. Find this divisor.

50. A pane of glass 56 inches long and 33 inches wide is to be reduced to one-half its area by cutting from each side and end strips of the same breadth. How broad must these strips be?

51. A circular grass plat is surrounded by a paved walk 8 feet wide. The area of the walk is equal to the area of the grass plat. Find the radius of the grass plat.

52. A father provided in his will that 700 acres of land should be divided equally among his sons; but, one of the sons not surviving the father, each of the others received 35 acres more than he otherwise would have received. How many sons were there at first?

53. Divide a line whose length is a into two parts such that the whole line shall be to the longer part as the longer part is to the shorter. What are the results when $a = 10$?

NOTE. A line so divided is said to be divided in extreme and mean ratio.

54. A boat's crew can row $3\frac{1}{2}$ miles down a river and back again in $1\frac{2}{3}$ hours. If the river flows uniformly at the rate of 2 miles an hour, what is their rate of rowing in still water?

55. A boat's crew can row a miles down stream and back again in b hours. If the rate of the stream is given, what is the rate of rowing in still water? If the rate of rowing in still water is given, what is the rate of the stream?

56. Divide 45 into two parts such that the sum of the quotients found by dividing each part by the other is $3\frac{11}{14}$.

57. Divide a into two parts such that the sum of the quotients found by dividing each part by the other is m.

58. Two fields have the same area; but one is an exact

square, and the other is a rectangle. Find the ratio of the length to the breadth of the rectangle if the distance around it is twice (m times) as great as the distance around the square.

59. Find two numbers, in the ratio of 4 to 5, such that, if 6 be added to the larger and 1 to the smaller, the square roots of the results shall differ by 1.

60. Four numbers are in proportion. The sum of the means is 19, the sum of the extremes is 21, and the sum of the squares of the four numbers is 442. Find these numbers.

61. The distance around a certain right-angled triangle is 30 feet, and its area is 30 square feet. Find each side.

62. The hypothenuse of a right-angled triangle is 60 feet, and its area is 864 square feet. Find the sides.

Generalize, putting h for the hypothenuse, and A for the area.

63. There are two numbers whose sum multiplied by the less is equal to four times the greater, but whose sum multiplied by the greater is equal to nine times the less. What are the numbers?

64. A square field and a rectangular field have the same area. The latter is three times as long as it is wide, and the distance around it exceeds the distance around the former by 128 rods. Find the dimensions of each field.